Towards NNLO QCD calculations: the singular behaviour of tree-level QCD amplitudes at $\mathcal{O}(\alpha_S^2)$

Massimiliano Grazzini

(S. Catani, M.G.)

Padova, ECFA meeting, May 7 2000

Outline

- Introduction: QCD calculations at NLO
- QCD calculations at NNLO
- IR structure at $\mathcal{O}(\alpha_S^2)$: state of the art
- Singularities in tree-level amplitudes at $\mathcal{O}(\alpha_S^2)$:
  - Collinear behaviour
  - Soft-Collinear behaviour
  - Soft behaviour
- Summary

QCD calculations at NLO

Difficulties: one has to consider virtual and real corrections which are affected by different kinds of singularities

UV singularities in virtual contributions ⇒ removed by renormalization

Soft (low momentum) and Collinear (small angle) singularities are present both in virtual and real corrections

The simplest calculations are those of fully inclusive quantities. In this case one has to integrate over the whole phase space, thus it is possible to add virtual and real contributions and cancel singularities.

In the case of less inclusive quantities the situation is much more involved

Real and virtual contributions have a different number of final state partons and so must be integrated separately

Soft and collinear divergences must be regularized first, usually by working in D dimensions

Virtual and real contributions should be calculated independently and the divergences cancelled

However the analytic continuation complicates the algebra and prevents a straightforward implementation of numerical integration techniques

Today efficient techniques have been set up to face these difficulties

These techniques combine analytic calculations with numerical integration

There are two kinds of such methods:

- Phase-space slicing (cone) method

K.Fabricius, G.Kramer, G.Schierholz, I.Schmitt (1981)
W.T.Giele, E.W.N.Glover (1992)
W.T.Giele, E.W.N.Glover, D.A.Kosower (1993)

- Subtraction method

R.K.Ellis, D.A.Ross, A.E.Terrano (1981)
Z.Kunszt, D.E.Soper (1992)

Both methods have been first used to calculate three jet cross-sections at NLO in e^+e^- collisions.

Then they have been adapted to deal with other processes. Only recently it became clear that both methods are generalizable in a process-independent manner

Key observation: the singular part of QCD matrix elements for real emission can be singled out in a general way by using factorization properties in the soft and collinear limits

Recently a completely numerical method has been proposed

D.E.Soper (1998)

Subtraction method:

S.Frixione, Z.Kunszt, A.Signer (1995)
S.Catani, M.H.Seymour (1996)
Z.Nagy and Z.Trocsanyi (1996)

General idea:

$$\begin{aligned} d\sigma^{NLO} &= d\sigma^R + d\sigma^V \\ &= \left[d\sigma^R - d\sigma^A\right] + d\sigma^A + d\sigma^V \end{aligned}$$

$d\sigma^A$ local counterterm for $d\sigma_R$ with the same singularity structure

The first term is now regular and can be integrated numerically in $D = 4$

If one is able to carry out the integration over the one-parton subspace of $d\sigma_A$ analytically ⇒ one cancels the singularities and do the remaining integral numerically

$$\sigma^{NLO} = \int_{m+1} \left[d\sigma^R - d\sigma^A\right]_{\epsilon=0} + \int_m \left(d\sigma^V + \int_1 d\sigma^A\right)_{\epsilon=0}$$

To implement the method one needs the local behaviour of the matrix element in the singular (soft and collinear) limits

The method can be made process-independent thanks to the universality properties of QCD amplitudes in these limits

Why going to NNLO ?

- Error estimates: if reliable PT starts at NLO reliable error estimates start at NNLO
- High precision measurements demand better theoretical accuracy:
  - NNLO corrections to the three jet rate would reduce the error on $\alpha_S(M_Z)$ from event shape analysis to the $2-3\%$ level
- For a next generation e^+e^- collider:
  - smaller α_S but much smaller power corrections $\Rightarrow$ NNLO corrections could be even more relevant
- Search for new physics: SM background has to be known very well

QCD calculations at NNLO

They are still a challenge in practise

To do:

- Evaluation of two-loop matrix elements:
  - recently important two-loop diagrams have been computed

    V.A Smirnov, O.L.Veretin (1999), Tausk (1999)
  - the 4g amplitude with maximal helicity violation has been computed

    Z. Bern, L. Dixon, D.A. Kosower (2000)
- Matching virtual and real contributions to cancel IR singularities

What to do?

A basic ingredient to achieve the feasibility of NLO calculations has been the complete understanding of the factorization properties of tree level and one-loop amplitudes in the soft and collinear limits

⇓

A similar understanding at NNLO would be very important

IR structure at $\mathcal{O}(\alpha_S)$: State of the art

- Two loop amplitudes:
  - The universal structure of the coefficients of the $1/\epsilon^4$, $1/\epsilon^3$, $1/\epsilon^2$ poles in two-loop amplitudes has been determined

    S.Catani (1998)
  - the general structure of $1/\epsilon$ poles must be determined
- One loop amplitudes with real emission:
  - The collinear limit is known

    Z.Bern, L.Dixon, D.C.Dumbar, D.A.Kosower (1994)
    Z.Bern, G.Chalmers, L.Dixon, D.A.Kosower (1994)
    Z.Bern, G.Chalmers (1995)
    Z.Bern, V.Del Duca, C.Schmidt (1998)
    Z.Bern, V.Del Duca, W.B.Kilgore, C.Schmidt (1999)
    D.A.Kosower, P.Uwer (1999)
  - The soft limit is known

    Z.Bern, V.Del Duca, C.Schmidt (1998)
    Z.Bern, V.Del Duca, W.B.Kilgore, C.Schmidt (1999)
- Double real emission corrections:
  - Collinear limit

    J.M.Campbell, E.W.N.Glover (1997)
    S. Catani, M.G. (1998)
  - Soft-collinear limit

    J.M.Campbell, E.W.N.Glover (1997)
    S.Catani, M.G (1999)
  - Soft limit

    F.A.Berends, W.T.Giele (1989), S.Catani (1992)
    S. Catani, M.G (1999)

Tree level amplitudes: singularities at $\mathcal{O}(\alpha_S^2)$

In order to study this problem there are in principle two strategies:

- Assume factorization and directly extract from available matrix elements the new universal functions that control the singularities in the various limits

J.M.Campbell, E.W.N.Glover (1997)

- Perform a general calculation

We have followed the second strategy by using power counting arguments and the universality properties of soft and collinear singularities

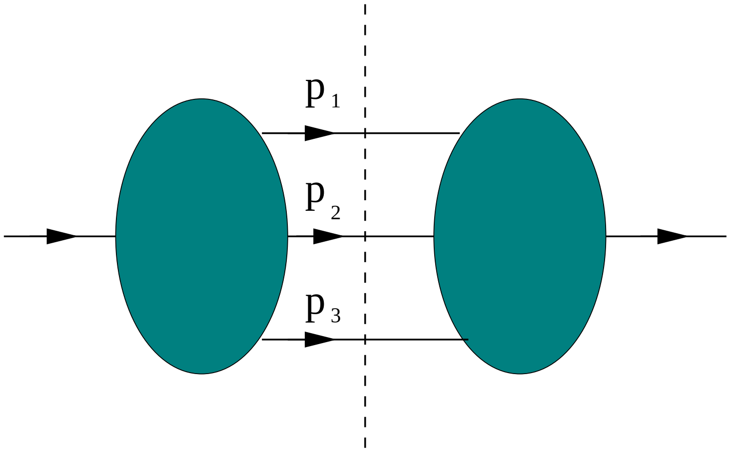
Since to implement the subtraction method one needs to identify and subtract a local counterterm, we want to keep fully into account azimuthal-correlations

The collinear behaviour

At this order there are two possible collinear limits:

- The first is when two pairs of partons become independently parallel and is treated with a straightforward iteration of the AP kernel
- The second is when three parton momenta become simultaneously parallel

In order to study this limit we have calculated process-independent Feynman graphs in a physical gauge



$\Rightarrow$ Interference contributions may be neglected

We worked in $D = 4 - 2\epsilon$ dimensions with $D - 2$ helicity states for the gluon and 2 for massless quarks (Conventional-Dimensional-Regularization scheme)

The three parton collinear limit is defined as

$$p_i^\mu = x_i p^\mu + k_{\perp i}^\mu - \frac{k_{\perp i}^2}{x_i}\frac{n^\mu}{2p\cdot n}\,, \qquad i = 1,2,3$$

$$s_{123} = (p_1+p_2+p_3)^2 \qquad s_{ij} = (p_i+p_j)^2$$

We find:

$$\boxed{|\mathcal{M}_{a_1,a_2,a_3,..}(p_1,p_2,p_3,..)|^2 \simeq \frac{4}{s_{123}^2}(4\pi\mu^{2\epsilon}\alpha_S)^2 \mathcal{T}_{a,\ldots}^{ss'}(p,\ldots)\hat{P}_{a_1a_2a_3}^{ss'}}$$

where

$$\mathcal{T}_{a_1,..}^{s_1s_1'}(p_1,..) \equiv \sum_{\text{spins}\neq s_1,s_1'}\sum_{\text{colours}} \mathcal{M}_{a_1,a_2,..}^{c_1,c_2,\ldots;s_1,s_2,..}(p_1,p_2,\ldots)$$
$$\times\left[\mathcal{M}_{a_1,a_2,\ldots}^{c_1,c_2,..;s_1',s_2,..}(p_1,p_2,\ldots)\right]^\dagger$$

and the three parton splitting functions $\hat{P}_{a_1a_2a_3}$ are universal and generalize the AP splitting functions

Spin correlations produced in the collinear splitting are universally taken into account by these splitting functions

By Lorentz invariance $\hat{P}_{a_1a_2a_3}$ may depend only on s_{ij}/s_{123} and on the combinations

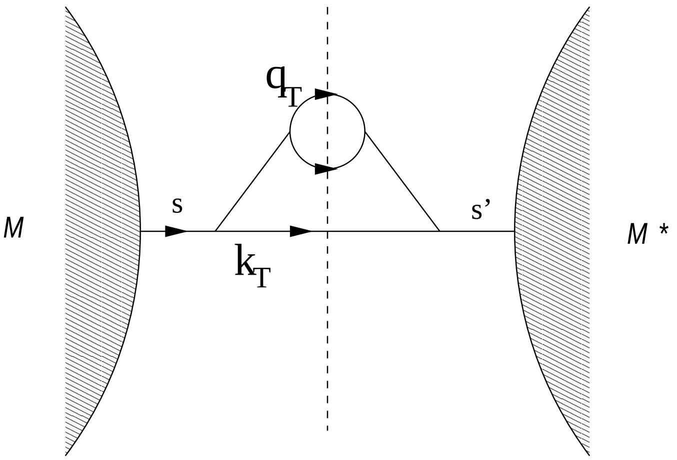
$$z_i = \frac{x_i}{\sum_{j=1}^3 x_j}$$

$$\widetilde{k}_i^\mu = k_{\perp i}^\mu - \frac{x_i}{\sum_{k=1}^3 x_k}\sum_{j=1}^3 k_{\perp j}^\mu$$

which automatically satisfy the constraints

$$\sum_{i=1}^3 z_i = 1 \qquad \sum_{i=1}^3 \widetilde{k}_i = 0$$

Let us consider the structure of the singularities arising in the three-parton collinear limit



$$\frac{1}{q_T^2(k_T^2+q_T^2)^2}\left(k_T^2\ A_1(k_T,q_T)+q_T^2\ A_2(k_T,q_T)+k_T\cdot q_T\ A_3(k_T,q_T)\right)$$

$$\frac{1}{q_T^2(k_T^2+q_T^2)}A_4(k_T,q_T)$$

$$\frac{1}{(k_T^2+q_T^2)^2}A_5(k_T,q_T)$$

$$\frac{1}{q_T^2}$$

As far as poles of order $1/ss'$ are concerned one can set

$k_T, q_T \to 0$ in the functions A_i and in the amplitude

$\Downarrow$

Poles of order $1/ss'$ are universal

Poles of order $1/s$ are in general process dependent

- The calculation: quark as parent parton

  The splitting-processes to be considered are

  - $q \to \bar{q}'_1 + q'_2 + q_3 \quad (\bar{q} \to \bar{q}'_1 + q'_2 + \bar{q}_3)$
  - $q \to \bar{q}_1 + q_2 + q_3 \quad (\bar{q} \to \bar{q}_1 + q_2 + \bar{q}_3)$
  - $q \to g_1 + g_2 + q_3 \quad (\bar{q} \to g_1 + g_2 + \bar{q}_3)$

  We find that in the collinear limit only one spin structure survives: ⇒ spin correlations are absent

  This is analogous to what happens at $\mathcal{O}(\alpha_S)$

  It is a consequence of helicity conservation in the quark-gluon coupling

- The calculation: gluon as parent parton

  The splitting processes to be considered are:

  - $g \to g_1 + q_2 + \bar{q}_3$
  - $g \to g_1 + g_2 + g_3$

  In this case spin correlations are highly non trivial

Properties of $\widehat{P}^{ss'}_{a_1a_2a_3}$

$N=1$ SUSY identity

At tree level QCD become supersymmetric by setting $C_F = C_A = 2T_R$

However in the regularization scheme we have worked SUSY is broken because the number of polarizations is 2 for the quark and $2-2\epsilon$ for the gluon

$\Rightarrow$ In the limit $\epsilon \to 0$ AP splitting functions satisfy an $N=1$ SUSY identity:

$$\widehat{P}_{q_1g_2} + (1 \leftrightarrow 2) = 2\widehat{P}_{q_1\bar{q}_2} + \widehat{P}_{g_1g_2}$$

Our splitting functions fulfil in the same limit a similar identity:

$$\left[\widehat{P}_{\bar{q}_1q_2q_3} + (1 \leftrightarrow 2) + (1 \leftrightarrow 3)\right] + \left[\widehat{P}_{q_3g_1g_2} + (1 \leftrightarrow 3) + (2 \leftrightarrow 3)\right]$$
$$= P_{g_1g_2g_3} + \left[\widehat{P}_{g_3\bar{q}_1q_2} + 5 \text{ permutations}\right]$$

Strong-ordered limit

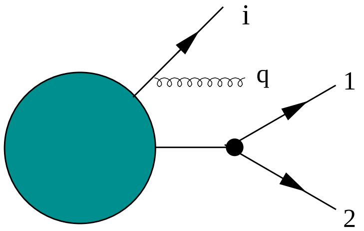
A further check of the results is provided by the strong-ordered limit

In this limit the particles become collinear sequentially and $\widehat{P}^{ss'}_{a_1a_2a_3}$ factorize into the product of two AP splitting functions

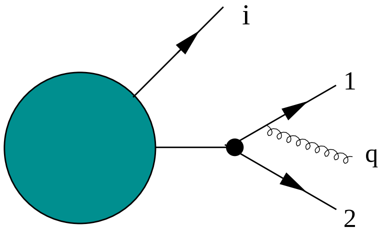
The soft-collinear behaviour

This limit is defined when the momentum q of a gluon becomes soft and at the same time two parton momenta p_1 and p_2 become collinear

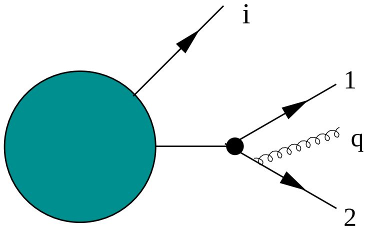
The soft limit can be studied by using the soft gluon insertion technique



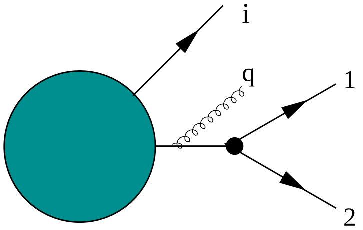
(a)



(b)



(c)



(d)

We find that in the collinear limit there is a cancellation between the diagrams (b) and (c) such that we can write a factorization formula

$$|\mathcal{M}_{g,a_1,a_2,\ldots,a_n}|^2 \simeq -\frac{2}{s_{12}} \, (4\pi\mu^{2\epsilon}\alpha_S)^2$$

$$\times \; \langle \, \mathcal{M}_{a,\ldots,a_n}(p) \, | \, \hat{\boldsymbol{P}}_{a_1 a_2} \, \left[\mathbf{J}^{\dagger}_{(12)\mu}(q) \, \mathbf{J}^{\mu}_{(12)}(q) \right] | \, \mathcal{M}_{a,\ldots,a_n}(p) \, \rangle$$

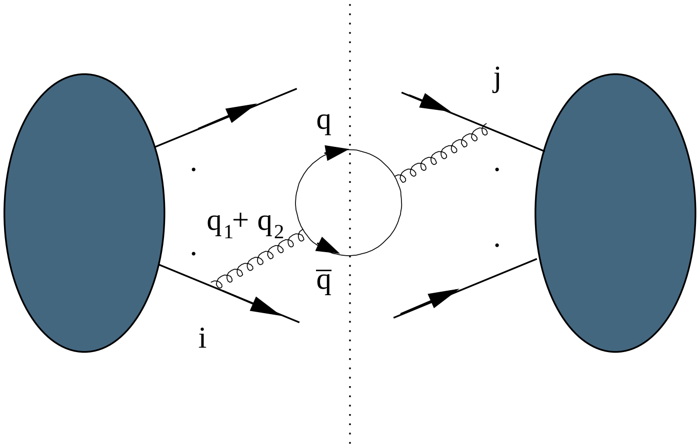
where

$$\mathbf{J}^{\mu}_{(12)}(q) \simeq \sum_{i=3}^{n} \mathbf{T}_i \, \frac{p_i^\mu}{p_i \cdot q} + \mathbf{T}_{(12)} \frac{p_1^\mu + p_2^\mu}{(p_1 + p_2) \cdot q}$$

⇒ coherence of soft gluon emission by the collinear pair

The soft behaviour: emission of a $q\bar{q}$ pair

In the limit in which the momenta q_1 and q_2 of a $q\bar{q}$ pair become soft there is a new subleading singularity arising from the propagator of the parent gluon



a factorization formula can be derived by using the soft-gluon insertion technique

$$|\mathcal{M}_{q_1,\bar{q}_2,a_3,\dots}(p_1,p_2,p_3,\dots)|^2 \simeq \frac{4}{(q_1+q_2)^4}\,T_R\,(4\pi\mu^{2\epsilon}\alpha_S)^2$$
$$\times \sum_{ij} \mathcal{I}_{ij}\,\langle\,\mathcal{M}_{a_3,\dots,a_n}\,|\mathbf{T}_i\mathbf{T}_j|\,\mathcal{M}_{a_3,\dots,a_n}\,\rangle$$

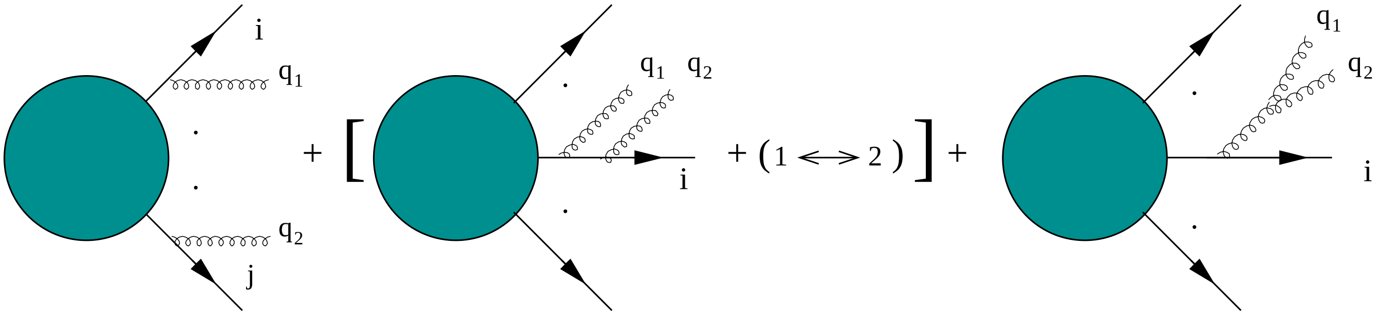
where

$$\mathcal{I}_{ij} = \frac{q_1\,p_i\,q_2\,p_j + q_1\,p_j\,q_2\,p_i - q_1\,q_2\,p_i\,p_j}{p_i(q_1+q_2)\,p_j(q_1+q_2)}$$

The soft behaviour: emission of two gluons

F.A.Berends, W.T.Giele (1989)
S.Catani (1992)

The limit in which the momenta q_1 and q_2 of two gluons become soft can be studied again by using the soft gluon insertion technique



(a) (b) (c)

The amplitude factorizes as

$$|\,\mathcal{M}^{\alpha_1\alpha_2}(q_1,q_2,p_i)\rangle \simeq g_S^2\mu^{2\epsilon}\,\epsilon^{\alpha_1}_{\mu_1}(q_1)\,\epsilon^{\alpha_2}_{\mu_2}(q_2)\,\mathbf{J}^{\mu_1\mu_2}(q_1,q_2)\,|\,\mathcal{M}(p_i)\rangle$$

where

$$\begin{aligned}
J^{\mu_1\mu_2}_{a_1a_2}(q_1,q_2) = {} & \sum_{i\neq j} T_i^{a_1}\frac{p_i^{\mu_1}}{p_iq_1}\,T_j^{a_2}\frac{p_j^{\mu_2}}{p_jq_2} \\
& + \sum_i \Bigg[\left(\delta^{a_1a}\,T_i^{a_2}\,\frac{p_i^{\mu_2}}{p_iq_2} - if_{a_2a_1a}\frac{q_1^{\mu_2}}{q_1q_2}\right)\,T_i^a\,\frac{p_i^{\mu_1}}{p_i(q_1+q_2)} \\
& + \left(\delta^{a_2a}\,T_i^{a_1}\,\frac{p_i^{\mu_1}}{p_iq_1} - if_{a_1a_2a}\frac{q_2^{\mu_1}}{q_1q_2}\right)\,T_i^a\,\frac{p_i^{\mu_2}}{p_i(q_1+q_2)} \\
& + \frac{1}{2}if^{aa_1a_2}\,T_i^a\,\frac{g^{\mu_1\mu_2}}{q_1q_2}\,\frac{p_i(q_2-q_1)}{p_i(q_2+q_1)}\Bigg]
\end{aligned}$$

The double soft limit of $|\mathcal{M}_{g,g,a_1,\ldots,a_n}(q_1,q_2,,p_1,\ldots,p_n)|^2$ is controlled by the square of the two-gluon current

We find

$$\left[J^{a_1a_2}_{\mu\rho}(q_1,q_2)\right]^\dagger d^{\mu\nu}(q_1)\, d^{\rho\sigma}(q_2)\; J^{a_1a_2}_{\nu\sigma}(q_1,q_2) = \frac{1}{2}\left\{\boldsymbol{J}^2(q_1)\,,\boldsymbol{J}^2(q_2)\right\}$$
$$-\,C_A\sum_{i,j=1}^{n}\boldsymbol{T}_i\cdot\boldsymbol{T}_j\;\mathcal{S}_{ij}(q_1,q_2) + \text{ gauge dependent terms}$$

where

$$\begin{aligned}\mathcal{S}_{ij}(q_1,q_2) &= \frac{(1-\epsilon)}{(q_1\,q_2)^2}\,\frac{p_i\,q_1\;p_j\,q_2 + p_i\,q_2\;p_j\,q_1}{p_i(q_1+q_2)\;p_j(q_1+q_2)}\\ &\quad - \frac{(p_i\,p_j)^2}{2p_i\,q_1\;p_jq_2\;p_i\,q_2\;p_jq_1}\left[2 - \frac{p_i\,q_1\;p_j\,q_2 + p_i\,q_2\;p_j\,q_1}{p_i(q_1+q_2)\;p_j(q_1+q_2)}\right]\\ &\quad + \frac{p_ip_j}{2q_1q_2}\left[\frac{2}{p_i\,q_1\;p_j\,q_2} + \frac{2}{p_j\,q_1\;p_i\,q_2}\right.\\ &\quad \left. - \frac{1}{p_i(q_1+q_2)\;p_j(q_1+q_2)}\left(4 + \frac{(p_i\,q_1\;p_j\,q_2 + p_i\,q_2\;p_j\,q_1)^2}{p_i\,q_1\;p_j\,q_2\;p_i\,q_2\;p_j\,q_1}\right)\right]\end{aligned}$$

⇒ No colour correlations in the double soft limit of four and five parton amplitudes

Summary

We have studied the behaviour of tree-level QCD amplitudes in the various soft and collinear limits at $\mathcal{O}(\alpha_S^2)$:

- In the triple collinear limit the singularities are controlled by process-independent splitting functions which generalize ordinary AP splitting functions and fully take into account azimuthal correlations

- In the soft-collinear limit the singularities are controlled by a factorization formula written only in terms of eikonal factors and AP splitting functions

- In the limit in which a $q\bar{q}$ pair become soft there is a new singularity but a factorization formula can be written by using a simple insertion factor

- The limit in which two gluons become soft is described in terms of a $\mathcal{O}(\alpha_S^2)$ generalization of the eikonal current
  We have obtained a compact expression for its square, that shows the absence of color correlations in four- and five-parton amplitudes

These results are one of the necessary ingredients to extend QCD predictions to NNLO