With:

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Outline

1. Empirical features of asset returns
2. The model
3. Calibration of the model
4. S&P500 index
5. Conclusions
Detrended asset logarithmic return

\[ X_t := \ln S_t - \ln S_{t-1} - \mu \quad (t = 1, 2, \ldots), \]

\( S_t \) being the price of a financial asset.

\textit{Stylized facts} are statistical features common to many assets:

- Heavy tails of returns’ distribution
- Absence of linear autocorrelations
- Slow decay of autocorrelation in absolute returns
- Anomalous scaling and multiscaling of aggregated returns
- Leverage effect
- ...
Scaling properties

The aggregated return over a period $t$ is $X_1 + \ldots + X_t$.

**Simple scaling** is present if

$$X_1 + \ldots + X_t \overset{dist}{=} t^H X_1.$$  \hspace{1cm} (1)

**Normal scaling** when $X_1$ is a Gaussian variable and $H = 1/2$, **anomalous scaling** otherwise.

**Multiscaling** is present if

$$\mathbb{E}[|X_1 + \ldots + X_t|^q] = t^{q H_q} \mathbb{E}[|X_1|^q]$$ \hspace{1cm} (2)

for existing moments with $H_q$ non–trivial function of $q$. 
Baldovin–Stella suggestion (PNAS, 2007)

Anomalous scaling with $X_1$ non-Gaussian and/or $H \neq 1/2$:

$$
\begin{align*}
X_t &:= a_t \ Y_t; \\
Y_t &:= \sum G_t,
\end{align*}
$$

with

- a modulating factor $a_t := \sqrt{t^{2H} - (t - 1)^{2H}}$
- $\{G_t\}_{t=1}^{\infty}$ i.i.d. standard normal variables
- $\Sigma$ a positive random variable independent of $\{G_t\}_{t=1}^{\infty}$.

Model inspired by an inverse renormalization–group strategy.
Strengths and weaknesses

Merits:

- **simple scaling** with any $H$ is obtained
  $$(X_1 + \ldots + X_t \overset{\text{dist}}{=} t^H X_1)$$
- The distribution of $X_t$’s belongs to the **rich class** of variance–Gaussian mixtures:

  \[
P[X_1 = x] = \int_0^\infty d\sigma \, \rho(\sigma) \frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi}\sigma^2},
\]

  $\rho$ being the probability density of $\Sigma$

Important **missing** properties:

- **stationarity** of the observed process
- **ergodicity**
Recover stationarity

Time restart mechanism of the modulating factor $a_t$:

- set $l_{t+1} = 1$ with probability $0 < \nu \leq 1$, otherwise $l_{t+1} = l_t + 1$
- in $a_t$, substitute $t$ with the random time $l_t$

$$X_t := a_{l_t} Y_t$$

$\{X_t\}_{t=1}^{\infty}$ is a stationary process if

- $\{l_t\}_{t=1}^{\infty}$ is independent of $\{Y_t\}_{t=1}^{\infty}$
- $\mathbb{P}[l_1 = i] := \nu(1 - \nu)^{i-1}$

Interpretation: time restarts may reflect exogenous events.
Recover ergodicity

Fix a memory order $M$.

- Distribute $Y_1, \ldots, Y_{M+1}$ as before:

$$P[Y_1 = y_1, \ldots, Y_{M+1} = y_{M+1}] := \int_0^\infty d\sigma \rho(\sigma) \prod_{t=1}^{M+1} \frac{e^{-y_t^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}.$$

- For $t > M + 1$, draw $Y_t$ according to the law

$$P[Y_t = y_t | Y_{t-1} = y_{t-1}, \ldots, Y_1 = y_1] := P[Y_{M+1} = y_t | Y_M = y_{t-1}, \ldots, Y_1 = y_{t-M}]$$

Stationarity is preserved and ergodicity is recovered.
Long and short memory

\[ X_t := a_{l_t} Y_t \]

is stationary and ergodic with

- \( \{a_{l_t}\}_{t=1}^{\infty} \) short memory (1) process
  - (exogenous events)
- \( \{Y_t\}_{t=1}^{\infty} \) long memory (M) process
  - (endogenous dynamics – volatility clustering)

Simple scaling promoted to approximated multiscaling!
A convenient choice of the density $\rho$

Weigh $\sigma^2$ according to an inverse–gamma distribution:

$$\rho(\sigma) := \frac{2^{1-\alpha/2}}{\Gamma(\alpha/2)} \frac{\beta^{\alpha}}{\sigma^{\alpha+1}} e^{-\beta^2/2\sigma^2}$$

with $\alpha$ and $\beta$ form and scale parameters.

- $X_t$’s display heavy tails with tail index $\alpha$
- $\beta$ fixes the scale of $X_t$’s fluctuations
- $\{ Y_t \}_{t=1}^{\infty}$ becomes a pure ARCH process of order $M$
- $X_t = a_t Y_t$ becomes a SWARCH model (Hamilton & Susmel, 1994)
Model parameters

Five parameters within the SWARCH prescription:

- $M \geq 1$ (memory of “endogenous component”)
- $\alpha > 0$ (tail index of returns’ distribution)
- $\beta > 0$ (fixes the scale of returns’ fluctuations)
- $0 < \nu \leq 1$ (time restart probability)
- $0 < D \leq 1/2$ (quantifies the effects of restarts):

$$a_i := \sqrt{i^{2D} - (i - 1)^{2D}}$$
Calibration via moment optimization

Compare theoretical and empirical moments.

Given $M$, estimate $\alpha$, $\nu$, and $D$ with

$$
\begin{align*}
m_q(t) & := \frac{\mathbb{E}[|X_1 + \ldots + X_t|^q]}{\mathbb{E}[|X_1|^q]} ;
\end{align*}
$$

$$
\begin{align*}
r_q(t) & := \frac{\mathbb{E}[|X_1 X_t|^q] - \mathbb{E}[|X_1|^q]^2}{\mathbb{E}[|X_1|^{2q}] - \mathbb{E}[|X_1|^q]^2} ,
\end{align*}
$$

for all $t \leq M$.

Estimate $\beta$ with $\mathbb{E}[|X_1|^q]$. 


Calibration for S&P500


Calibration with only moment order $q = 1$.

- $M = 21, \alpha = 4.0, \beta = 0.04, \nu = 0.030, D = 0.21$
- $M = 42, \alpha = 4.5, \beta = 0.07, \nu = 0.011, D = 0.19$
- $M = 63, \alpha = 5.5, \beta = 0.14, \nu = 0.004, D = 0.16$
Fit performances
Multiscaling features

The graph shows the scaling behavior of the S&P index from 1950-2010, with distinct periods marked from 1950-1970, 1970-1990, 1990-2010, and 1950-2010 (*). The data points are plotted on a log-log scale, indicating the superlinear scaling of the index returns with respect to the scaling parameter $q$. The different colors and markers represent different periods, illustrating how the scaling behavior changes over time.
Returns’ distribution
Endogenous volatility

Can we distinguish long– and short–memory contributions to volatility?

- Try to identify time restarts in a time series \( \{\bar{x}_t\}_{t=1}^T \) studying
  \[
P[l_t = 1 | X_n = \bar{x}_n, |n - t| \leq \tau] \quad (t = 1, 2, \ldots, T)
  \]

- Isolate the endogenous path \( \{\bar{y}_t\}_{t=1}^T \)

- Sample volatility on time horizon \( t \):
  \[
  \sqrt{\frac{1}{t} \sum_{n=1}^{t} Y_n^2}
  \]
Calibration with $M = 63$ and time horizon $t = M$
Scaling symmetry allows to construct a discrete–time stochastic model for asset dynamics which is

- a combination of a short–memory modulating component and a long–memory endogenous component
- rich (multiscaling, heavy tails, slow autocorrelation decay)
- parsimonious (five parameters)
- easy to calibrate
- useful in applications (derivative pricing)

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