

Enstrophy dissipation in two-dimensional turbulence

Marco Baiesi and Christian Maes*

Instituut voor Theoretische Fysica, K.U.Leuven, Celestijnenlaan 200D, B-3001, Belgium

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Insight into the problem of two-dimensional turbulence can be obtained by an analogy with a heat conduction network. It allows the identification of an entropy function associated with the enstrophy dissipation and that fluctuates around a positive (mean) value. While the corresponding enstrophy network is highly nonlocal, the direction of the enstrophy current follows from the Second Law of Thermodynamics. An essential parameter is the ratio $T_k \equiv \gamma_k / (\nu k^2)$ of the intensity of driving $\gamma_k > 0$ as a function of wave number k , to the dissipation strength νk^2 , where ν is the viscosity. The enstrophy current flows from higher to lower values of T_k , similar to a heat current from higher to lower temperature. Our probabilistic analysis of the enstrophy dissipation and the analogy with heat conduction thus complements and visualizes the more traditional spectral arguments for the direct enstrophy cascade. We also show a fluctuation symmetry in the distribution of the total entropy production which relates the probabilities of direct and inverse enstrophy cascades.

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I. INTRODUCTION

Three-dimensional turbulence displays an inertial range, in which energy is transferred from the spatial scales at which it is introduced into the system down to small scales, where it is finally dissipated by viscous forces. The standard picture of turbulence in two dimensions is qualitatively different. Following the pioneering works of Kraichnan [1,2], Leith [3], and Batchelor [4] (KLB) on the two-dimensional Navier-Stokes equation for the fluid velocity field, one expects an inverse energy cascade from the forcing scales to large scales, and simultaneously a direct enstrophy cascade from the forcing scales to small scales. The enstrophy is the variance of the vorticity; namely, the ensemble average of the squared curl of the velocity.

Two-dimensional turbulence has been a very active area of theoretical, numerical, and experimental investigation [5,6], not only as an easier test case but also relevant to certain real quasi-two-dimensional situations. Examples include oceanic currents and atmospheric and geophysical flows [7], but two-dimensional flow is also realized in laboratory situations [5,6]. However, the picture of two-dimensional turbulence remains not fully understood and in fact, there are some limitations to the above classical KLB scenario. For example, the standard enstrophy cascade disappears when considering a bounded domain where only a monoscale forcing is applied [8]. Moreover, the mechanism of the direct enstrophy cascade and the determining factor for the direction of the enstrophy current has not been fully understood as a consequence of a more general principle. There have been recent clarifications, going into the details of the physical mechanism (e.g., Refs. [9,10]), but it seems interesting and natural to connect the situation also with better understood scenarios and to be able to see the enstrophy dissipation as the result of a more generally valid principle.

In the present paper, we address the issue of the enstrophy current and its direction. A very close analogy with a two-

dimensional heat conduction problem provides ingredients to understand the enstrophy cascade in its full qualitative behavior. It turns out, as will be shown later, that the stochastically driven Navier-Stokes equation for the vorticity can be mapped to a problem of heat conduction: at each wave number, k a thermal reservoir is attached with temperature $T_k = \gamma_k / (\nu k^2)$, where γ_k is the forcing strength and ν is the viscosity. From the Second Law of Thermodynamics, which will be derived in its detailed version, it follows that enstrophy is dissipated as heat flows: from higher to lower temperature, or here, when γ_k is peaked around some small mode k , from small to large wave numbers. In other words, the origin and the direction of the enstrophy flux is simply and directly a consequence of the Second Law applied to the enstrophy. We also go beyond the study of the average enstrophy current and discuss a symmetry in its fluctuations. That estimates the probability of going backwards; i.e., the probability of an inverse enstrophy cascade. At the same time, we obtain a steady state fluctuation theorem in the context of turbulence.

In the next section, we start by reminding the reader of the standard picture of two-dimensional turbulence. In Sec. III we present the analogy with heat conduction. From it follows the final analysis of the enstrophy dissipation in Sec. IV. The main general consequences and conclusions are discussed in Sec. V.

The paper will describe the arguments and analog in a formal way, avoiding, however, a fully rigorous mathematical analysis. The main goal is indeed to point out a useful picture and analogy that is sufficiently powerful to specify the enstrophy cascade. To add the mathematical details and hypotheses is not believed to be extremely difficult; only a few remarks are added to guide the mathematically inclined.

II. NAVIER-STOKES EQUATION

The Navier-Stokes (NS) equation [11,12] for the velocity field $\vec{u}(t, r)$ is

*Electronic address: christian.maes@fys.kuleuven.be

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} = \nu \Delta \vec{u} - \vec{\nabla} p + \vec{f}, \quad (2.1)$$

where p is the pressure, \vec{f} is the external force, and ν is the viscosity. This is supplemented by the incompressibility condition

$$\vec{\nabla} \cdot \vec{u} = 0,$$

and in our case by periodic boundary conditions for a finite spatial region V . Similar equations arise for the vorticity $\vec{\omega} \equiv \vec{\nabla} \times \vec{u}$, by taking the curl of (2.1)

$$\frac{\partial \vec{\omega}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{\omega} - (\vec{\omega} \cdot \vec{\nabla}) \vec{u} = \nu \Delta \vec{\omega} + \vec{g} \quad (2.2)$$

with $\vec{g} = \vec{\nabla} \times \vec{f}$. The energy of the system is given by the total kinetic energy

$$E = \int_V \frac{u^2}{2},$$

while the enstrophy is defined as

$$\Omega = \int_V \frac{\omega^2}{2}.$$

Its role will become clearer later on.

Consider Cartesian coordinates $r = (x_1, x_2, x_3) = (x, y, z)$; the two-dimensional case is conveniently represented by setting the third component of the velocity equal to zero: $\vec{u} = (u_1, u_2, 0)$. Therefore, $\vec{\omega} = (0, 0, \omega_3)$ is better represented by a pseudoscalar $\omega = \omega_3$. Equation (2.2) thus acts on a single component, and since \vec{u} and $\vec{\omega}$ are now perpendicular to each other, it is further simplified by the vanishing of the term $(\vec{\omega} \cdot \vec{\nabla}) \vec{u}$, as

$$\frac{\partial \omega}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \omega = \nu \Delta \omega + g, \quad (2.3)$$

where $g = \partial f_1 / \partial y - \partial f_2 / \partial x$. The pressure has disappeared but Eq. (2.3) is still nonlocal because $\vec{u} = \mathcal{K} \omega$ for some Biot-Savart kernel \mathcal{K} .

We take our system bounded in a rectangular domain, where it is useful to consider the Fourier transform

$$\omega_k = \frac{1}{2\pi} \int_V e^{ik \cdot r} \omega(r).$$

These modes satisfy $\bar{\omega}_k = \omega_{-k}$ which will always be understood.

Upon Fourier-transforming (2.3), we thus get, for $k \in \mathbb{Z}^2 \setminus \{0\}$,

$$\frac{\partial \omega_k}{\partial t} - F_k(\omega) = -\nu k^2 \omega_k + g_k, \quad (2.4)$$

where

$$F_k(\omega) = \sum_{j+\ell=k} \Phi_{j,\ell} \omega_j \omega_\ell \quad (2.5a)$$

with coefficients given by

$$\Phi_{j,\ell} \equiv \frac{j_2 \ell_1 - j_1 \ell_2}{4\pi} \left(\frac{1}{|j|^2} - \frac{1}{|\ell|^2} \right).$$

These $\Phi_{j,\ell} = 0$ if and only if either $j \parallel \ell$ or $|j| = |\ell|$. Alternatively,

$$F_k(\omega) = \sum_{0 \neq \ell \neq k} \phi_{k,\ell} \omega_\ell \omega_{k-\ell} \quad (2.5b)$$

with

$$\phi_{k,\ell} \equiv \frac{k_1 \ell_2 - \ell_1 k_2}{4\pi} \left(\frac{1}{|\ell|^2} - \frac{1}{|k-\ell|^2} \right) = \phi_{-k,-\ell}.$$

In that notation,

$$\phi_{k,\ell} + \phi_{k,k-\ell} = 0$$

represents the so called triad relation of [13]. Another alternative, which will be useful later, is

$$F_k(\omega) = -\frac{1}{2\pi} \sum_{\ell} \frac{k_1 \ell_2 - \ell_1 k_2}{|k-\ell|^2} \omega_\ell \omega_{k-\ell}. \quad (2.5c)$$

Finally, we must specify the forcing g_k . A translationally invariant and stationary turbulent state can be achieved by imposing a force that is homogeneous in space and time. A Gaussian random field with zero mean is the simplest example: in that case the force \vec{f} in (2.1) is a Gaussian noise that is white in time and colored in space, completely determined by its covariance

$$\langle f_i(s, r) f_j(t, r') \rangle = C_{ij}(r-r') \delta(t-s),$$

where $\partial_i C_{ij} = 0$ (incompressibility). Equation (2.4) then turns into the stochastically driven NS

$$d\omega_k(t) = -\nu k^2 \omega_k dt + F_k(\omega) dt + \sqrt{2\gamma_k} dW_k(t), \quad (2.6)$$

in which $dW_k = d\bar{W}_{-k}$ represents a standard Wiener process [14]. That driving pumps vorticity into the system at wave number k with intensity $\gamma_k \geq 0$, while the viscosity $\nu > 0$ enters in the first term on the right-hand side of (2.6) to dissipate the vorticity. Equation (2.6) is the starting point of our analysis.

A. Mathematical assumptions

The point of departure (2.6) is a stochastic differential equation to be understood in the Itô sense [14]. Solutions are Markov processes but note that they are infinite-dimensional. In general, the resulting diffusion is not elliptic because some γ_k can be made zero. That brings us to the problem of understanding the assumptions on the strengths γ_k and on the viscosity ν so that there is a unique invariant probability measure μ . A lot of mathematical work has been devoted to that problem in recent years. For example [15], if ν is sufficiently large as a function of $\sum \gamma_k^2$, then μ is unique. In addition, [16], μ is unique when there is $\kappa > 0$ so that $\gamma_k \sim |k|^{-\alpha}$ for some α and for every $|k| > \kappa$. Even [17–19] when $\gamma_k \neq 0$, for every $|k| \leq N$ where N is some number that depends on ν and on $\sum \gamma_k^2$, then μ is unique. We refer to Ref. [20] for even stronger and more recent results.

In what follows we simply assume, with no further ado, that μ is unique and has smooth local densities. Another assumption in the technical manipulations is to start from a finite-dimensional analysis. In other words, we choose a finite but arbitrary N and consider Eq. (2.6) only for $k^2 \leq N$ with $\gamma_k \neq 0$ there. That cutoff will take care of convergence problems in what follows and it allows us to speak of $\mu(\omega)$ as the density of μ with respect to the flat measure $d\omega$.

B. Euler equation

The vorticity and the corresponding enstrophy play an important role in two-dimensional turbulence because of the appearance of an extra conservation law. The Euler equation corresponding to (2.6) is

$$d\omega_k(t) = F_k(\omega)dt$$

[cf. (2.5a), (2.5b), and (2.5c)]. It is easy to see that

$$\sum_k \bar{\omega}_k F_k(\omega) = 0, \quad (2.7)$$

so that enstrophy is conserved:

$$\frac{d\Omega}{dt} = \sum_k \frac{1}{2} \frac{d|\omega_k|^2}{dt} = 0.$$

As a consequence, the enstrophy in (2.6) is changed by the injection (at rate γ_k) and the dissipation (with intensity ν) but is transported without dissipation over the various modes via the nonlinear and highly nonlocal terms in F_k . That invites the definition of various enstrophy currents.

C. Currents

The net enstrophy current J_k that leaves the system at wave number k is obtained from investigating the sources and sinks to the enstrophy. The total enstrophy dissipation over the time interval $[-\tau, \tau]$ is computed from

$$\Omega(\tau) = \frac{1}{2} \sum_k |\omega_k|^2(\tau)$$

and

$$\Omega(\tau) - \Omega(-\tau) = \sum_k \left[-\nu k^2 \int_{-\tau}^{\tau} |\omega_k|^2(t) dt + \sqrt{2\gamma_k} \operatorname{Re} \int_{-\tau}^{\tau} \bar{\omega}_k(t) \circ dW_k(t) \right],$$

where the last integral is in the Stratonovich sense [14], and Re stands for real part. The expression thus evaluates the change of enstrophy during a time interval $[-\tau, \tau]$ for a history $[\omega_k(t)]$ and for a realization of the noise $[dW_k(t)]$. It is therefore natural to set

$$J_k^{\text{out}} = \nu k^2 \int_{-\tau}^{\tau} |\omega_k|^2 dt,$$

$$J_k^{\text{in}} = \sqrt{2\gamma_k} \operatorname{Re} \int_{-\tau}^{\tau} \bar{\omega}_k \circ dW_k$$

as the current going, respectively, “out” and “in” the system at mode k with respect to the external enstrophy reservoir. The difference,

$$J_k \equiv J_k^{\text{out}} - J_k^{\text{in}},$$

is the net enstrophy current that leaves the system (enters the environment) at mode k .

On the other hand, the local conservation law reads

$$\begin{aligned} \frac{|\omega_k|^2(\tau)}{2} - \frac{|\omega_k|^2(-\tau)}{2} &= -\nu k^2 \int_{-\tau}^{\tau} |\omega_k|^2(t) dt + \operatorname{Re} \int_{-\tau}^{\tau} \bar{\omega}_k F_k(\omega) dt \\ &+ \sqrt{2\gamma_k} \operatorname{Re} \int_{-\tau}^{\tau} \bar{\omega}_k \circ dW_k(t) \\ &= -J_k^{\text{out}} + \sum_{\ell} J_{\ell k} + J_k^{\text{in}}, \end{aligned} \quad (2.8)$$

which defines

$$J_{\ell k} \equiv \frac{1}{2\pi} \operatorname{Re} \int_{-\tau}^{\tau} dt \frac{\ell_1 k_2 - k_1 \ell_2}{|\ell - k|^2} \omega_{-k} \omega_{k-\ell} \omega_{\ell}$$

the net current from mode ℓ to mode k [here we used (2.5c)]. Note the asymmetry $J_{k\ell} = -J_{\ell k}$.

As said before, the redistribution of enstrophy due to interactions between different modes globally does not change the total amount of enstrophy in the system: $\sum_k \sum_{\ell} J_{k\ell} = 0$ [see (2.7)].

One of the main problems for the cascade picture is to understand the direction of the flow of the $J_{k\ell}$. That is basically determined by the stationary $\langle J_k \rangle$. At the end of the paper, we also discuss its fluctuations.

D. Spectral distribution

Heuristically, the reason the enstrophy flows towards small scales (large wave number k) is because, at these small scales, the dissipative term $\nu \Delta \omega$ in (2.3) dominates over the advection term $(\vec{u} \cdot \vec{\nabla}) \omega$. A more refined argument, started by Kraichnan [1], derives the two-dimensional cascade picture (the so-called direct cascade for the enstrophy and the inverse cascade for the energy) by investigating the energy spectra. The Fourier spectrum of energy embodies the KLB picture by showing a power-law regime for each of the two cascades. Since the enstrophy spectrum is simply related to the energy one, from the inspection of the energy spectrum one can argue where energy and enstrophy are transferred or dissipated. In a way the cascade of enstrophy to small scales is the two-dimensional analog of the energy cascade in three dimensions. We skip here the details of that Kolmogorov-Kraichnan theory as they have been excellently reviewed by many. We refer to Ref. [21,22].

The purpose of the present paper is to try an alternative to that analysis by opening an analogy with heat conduction. It

is interesting that in this way a natural enstrophy dissipation function appears, the thermodynamic entropy production, as will be explained in Sec. IV.

III. FORMAL ANALOGY WITH HEAT DISSIPATION

Remember our starting equation (2.6). Let us first forget about the coupling between the various modes, so that the system is reduced to the stochastic dynamics

$$d\omega_k(t) = -\nu k^2 \omega_k(t) dt + \sqrt{2\gamma_k} dW_k(t) \quad (3.1)$$

describing an ensemble of uncoupled oscillators labeled by the wave number k . While in the original NS equation the viscosity represents an irreversible loss, here it balances reversibly with the stochastic forcing. The dynamics (3.1) has a reversible equilibrium measure

$$\mu^0(d\omega) = \prod_k \frac{e^{-|\omega_k|^2/2T_k}}{Z_k} d\omega_k d\bar{\omega}_k \quad (3.2)$$

that we will use as a reference.

The parameter $T_k \equiv \gamma_k / (\nu k^2)$ can be viewed as a kind of “temperature” of the reservoir attached to wave number k ; it is, of course, no physical temperature. Thus, our approach is different from previous attempts to use a thermodynamical formalism in turbulence, identifying variables like ω_k^2 with a temperature (see Ref. [23] and references therein).

The reversibility of the dynamics (3.1) is taken with the usual kinematical time reversal that reverses the sign of the velocity field: the dynamical time reversal of a history

$$\xi = (\omega(t), t \in [-\tau, \tau]) \quad (3.3)$$

in a given time interval $[-\tau, \tau]$ is

$$\Theta \xi = (-\omega(-t), t \in [-\tau, \tau]). \quad (3.4)$$

When we add the $F_k(\omega)$ to (3.1) to obtain (2.6) the oscillators become coupled, in fact, in a nonlinear and nonlocal way. That coupling does, however, preserve the enstrophy very much like a Hamiltonian coupling that conserves the energy. The picture that thus emerges is formally equivalent to a heat conduction network wherein the vertices of the network are represented by the modes k .

The Euler equation represents the conservative part of the time evolution. That is changed by the addition of the Langevin forces that represent “thermal” reservoirs at each of the k ; thus obtaining our equation (2.6). Observe that the “friction” depends on the “location” k of the oscillator. Standard thermodynamics then teaches us that there will be a “heat current” from higher to lower temperature. That “heat current” is in our present setup played by the enstrophy current. Hence, if the driving makes T_k a decreasing function of $|k|$, e.g., by having $\gamma_k \sim k^{-\alpha}$ for some $\alpha > 0$, then the enstrophy should be transported from small $|k|$ towards larger $|k|$. That picture will be detailed in the following sections.

The forward generator \mathcal{L}^+ corresponding to the Markov diffusion (2.6) can be split into a “conservative” and a “dissipative” part, $\mathcal{L}^+ = \mathcal{L}_c^+ + \mathcal{L}_d^+$, with

$$\mathcal{L}_c^+ \rho = - \sum_k F_k(\omega) \frac{\partial \rho}{\partial \omega_k} \quad (3.5a)$$

and

$$\mathcal{L}_d^+ \rho = \nu \sum_k k^2 \frac{\partial}{\partial \omega_k} (\omega_k \rho) + \sum_k \gamma_k \frac{\partial^2 \rho}{\partial \omega_k^2} = \sum_k \gamma_k \frac{\partial X_k}{\partial \omega_k}, \quad (3.5b)$$

where we made use of the shorthand

$$X_k \equiv e^{-\beta_k |\omega_k|^2/2} \frac{\partial}{\partial \omega_k} (e^{\beta_k |\omega_k|^2/2} \rho)$$

with $\beta_k \equiv \nu k^2 / \gamma_k$.

For the stationary measure μ , we have $\mathcal{L}^+ \mu = 0$, and in particular,

$$\langle F_k(\omega) \bar{\omega}_k \rangle - \nu k^2 \langle |\omega_k|^2 \rangle + \gamma_k = 0. \quad (3.6)$$

From now on we use that notation $\langle \cdot \rangle$ to denote a stationary average according to μ . Equation (3.6) gives, for every time interval $[-\tau, \tau]$,

$$\frac{1}{2\tau} \langle J_k \rangle = \nu k^2 (\langle |\omega_k|^2 \rangle - T_k). \quad (3.7)$$

This equation is the detailed enstrophy balance equation in stationarity; summing over k gives the somewhat more familiar

$$\nu \sum_k k^2 \langle |\omega_k|^2 \rangle = \sum_k \gamma_k,$$

but at the same time and as a new interpretation of (3.7), we recognize how the net current into the enstrophy reservoir at mode k is like a heat current into a thermal reservoir as determined by the difference between what now plays the role of a local kinetic temperature $\langle |\omega_k|^2 \rangle$ and the reservoir temperature T_k .

IV. ENSTROPY DISSIPATION

Continuing with the analogy above, a quantity is now brought to the forefront which we call the entropy current S . Since the net enstrophy current leaving the system at each mode k is $J_k = J_k^{\text{out}} - J_k^{\text{in}}$ and the corresponding “effective temperature” is T_k , we set

$$S \equiv \sum_k \frac{1}{T_k} J_k \quad (4.1)$$

as the variable entropy current. It is a function of the history (3.3) over $[-\tau, \tau]$. The entropy current S is the entropy production in the environment associated with the enstrophy dissipation; it is the usual sum over all dissipative currents divided by the respective temperatures. In the stationary state, the average $\langle S \rangle$ is the total change of the entropy in the universe over the time interval $[-\tau, \tau]$.

We will now show what is suggested thermodynamically by the previous analogy: S should measure the irreversibility

and $\langle S \rangle \geq 0$ as a consequence of the Second Law of Thermodynamics.

Remember that μ is the stationary measure of the NS dynamics (2.6); we denote by $\pi\mu$ its time reversal. In a given time interval $[-\tau, \tau]$, each history (3.3) is realized in the system with a probability that comes from the path-space measure $P_\mu^\tau(d\xi)$; i.e., the stationary Markov diffusion process associated with the stationary measure μ and the stochastic dynamics (2.6).

We compute the logarithmic density [see also (3.3) and (3.4)]

$$R \equiv \ln \frac{P_\mu^\tau}{P_{\pi\mu}^\tau \Theta} \quad (4.2)$$

as a measure of irreversibility. It gives the ratio between the probability of a history ξ and the probability of the time-reversed history $\Theta\xi$. We show that R coincides with S up to a temporal boundary term. Moreover, taking stationary averages, $\langle R \rangle = \langle S \rangle$. Since by construction $\langle R \rangle \geq 0$, it also follows that $\langle S \rangle \geq 0$.

To compute R , it is useful to compare the path-space measure with the reference path-space measure of the uncoupled case (3.1), denoted by $P_{\mu^0}^{0,\tau}$ (this is stationary and reversible). Thus, first we compute the action

$$A_\mu \equiv \ln \frac{P_\mu^\tau}{P_{\mu^0}^{0,\tau}}$$

and similarly $A_{\pi\mu} \circ \Theta$, to finally estimate (4.2) as the source of time-reversal breaking

$$R = A_\mu - A_{\pi\mu} \circ \Theta. \quad (4.3)$$

The comparison of the two measures P and P^0 is made by means of the Girsanov formula [14], obtaining

$$\begin{aligned} A_\mu = & \sum_k \frac{1}{2\gamma_k} \left\{ \int_{-\tau}^{\tau} \left[\nu k^2 \operatorname{Re}[\omega_k \bar{F}_k(\omega)] - \frac{1}{2} |F_k(\omega)|^2 \right] dt \right. \\ & \left. + \operatorname{Re} \left[\int_{-\tau}^{\tau} \bar{F}_k(\omega) d\omega_k \right] \right\} + \ln \mu[\omega(-\tau)] - \ln \mu^0[\omega(-\tau)]. \end{aligned}$$

Substituting $\Theta\xi$ gives

$$\begin{aligned} A_{\pi\mu} \circ \Theta = & \sum_k \frac{1}{2\gamma_k} \left\{ \int_{-\tau}^{\tau} \left[-\nu k^2 \operatorname{Re}[\omega_k \bar{F}_k(\omega)] - \frac{1}{2} |F_k(\omega)|^2 \right] dt \right. \\ & \left. + \operatorname{Re} \left[\int_{-\tau}^{\tau} \bar{F}_k(\omega) d\omega_k \right] \circ \Theta \right\} + \ln \mu[\omega(\tau)] - \ln \mu^0[\omega(\tau)]. \end{aligned}$$

Here, Itô stochastic integrals are performed and one should remember that these are themselves not time-reversal symmetric [14]. As an example for computing (4.3), we see that

$$\begin{aligned} & \operatorname{Re} \left[\int_{-\tau}^{\tau} \bar{F}_k(\omega) d\omega_k \right] - \operatorname{Re} \left[\int_{-\tau}^{\tau} \bar{F}_k(\omega) d\omega_k \right] \circ \Theta \\ &= \lim_{\Delta t \rightarrow 0} \operatorname{Re} \left[\sum_j \bar{F}_k(\omega(t_{j-1})) [\omega_k(t_j) - \omega_k(t_{j-1})] \right] \\ & \quad - \lim_{\Delta t \rightarrow 0} \operatorname{Re} \left[\sum_j \bar{F}_k(-\omega(t_j)) [-\omega_k(t_{j-1}) + \omega_k(t_j)] \right] \\ &= - \lim_{\Delta t \rightarrow 0} \operatorname{Re} \left[\sum_j \frac{\bar{F}_k(\omega(t_j)) - \bar{F}_k(\omega(t_{j-1}))}{\bar{\omega}_k(t_j) - \bar{\omega}_k(t_{j-1})} [\omega_k(t_j) - \omega_k(t_{j-1})]^2 \right] \\ &= - \operatorname{Re} \left[\int_{-\tau}^{\tau} \frac{\partial \bar{F}_k(\omega)}{\partial \bar{\omega}_k} dt \right] = 0, \end{aligned}$$

because $\partial \bar{F}_k / \partial \bar{\omega}_k = 0$ [see (2.5b)].

As a consequence, (4.3) becomes

$$R = S + \ln \mu[\omega(-\tau)] - \ln \mu[\omega(\tau)] \quad (4.4)$$

with

$$S = \sum_k \frac{1}{T_k} \left\{ \left[\frac{\omega_k^2(-\tau)}{2} - \frac{\omega_k^2(\tau)}{2} \right] + \int_{-\tau}^{\tau} \operatorname{Re}[\omega_k \bar{F}_k(\omega)] dt \right\}.$$

From (2.8), this expression coincides exactly with (4.1), as promised.

When, instead of the stationary μ , we had taken some initial density evolving as $\rho_t, t \in [-\tau, \tau]$, the analysis above would be essentially unchanged. In that case the source of irreversibility is

$$R = \sum_k \frac{1}{T_k} J_k + \ln \rho_{-\tau}(\omega_{-\tau}) - \ln \rho_\tau(\omega_\tau), \quad (4.5)$$

where the only difference with (4.4) resides in the last two terms, the temporal boundary

$$[-\ln \rho_\tau(\omega_\tau)] - [-\ln \rho_{-\tau}(\omega_{-\tau})].$$

A. Mean entropy production

From the definition (4.2) it directly follows that

$$\langle e^{-R} \rangle = 1$$

(it is essentially the normalization condition of the path-space measure $P_{\pi\mu}^\tau \Theta$). Hence, by a convexity inequality, the stationary enstrophy dissipation $\langle S \rangle = \langle R \rangle \geq 0$. We can, however, be more explicit concerning that point by deriving an expression for $\langle S \rangle$ that is explicitly non-negative. In fact, we will show that

$$\langle S \rangle = \sum_k \gamma_k \left\langle \left(\exp[-V_k(\omega)] \frac{\partial}{\partial \omega_k} \exp[V_k(\omega)] \right)^2 \right\rangle, \quad (4.6)$$

where

$$V_k(\omega) \equiv |\omega_k|^2 / (2T_k) + \ln \mu(\omega).$$

From (4.6), $\langle S \rangle > 0$ strictly, as we can only have

$$\frac{\partial}{\partial \omega_k} [e^{\beta_k |\omega_k|^2/2} \mu(\omega)] = 0$$

for all k when $\mu = \mu_0$ of (3.2).

Here comes the proof of (4.6). Denote by $\mathbb{E}_{\rho_{-\tau}}$ the expectation in the process $P_{\rho_{-\tau}}^\tau$ started from $\rho_{-\tau}$. We assume that at time τ the evolved measure is described by a density ρ_τ . We have (4.5), in expectation,

$$\mathbb{E}_{\rho_{-\tau}}[R] = \sum_k \beta_k \mathbb{E}_{\rho_{-\tau}}[J_k] + S(\rho_\tau) - S(\rho_{-\tau}), \quad (4.7)$$

where $S(\rho) \equiv -\int d\omega \rho(\omega) \ln \rho(\omega)$ is the Shannon entropy of the density ρ . Another formulation is

$$\mathbb{E}_{\rho_{-\tau}}[R] = \int_{-\tau}^\tau \dot{R}(t) dt$$

with, similar to (3.7),

$$\dot{R}(t) \equiv \nu \sum_k k^2 \beta_k \left[\int d\omega |\omega_k|^2 \rho_t(\omega) - \frac{1}{\beta_k} \right] + \frac{d}{dt} S(\rho_t). \quad (4.8)$$

The previous considerations thus identify the mean dissipation rate at time t (in the transient regime) with $\dot{R}(t)$.

To see the relation with (4.6), we start by evaluating the time-derivative of the Shannon entropy:

$$\frac{dS}{dt}(\rho) = - \int d\omega \frac{d\rho}{dt} \ln \rho = - \int d\omega (\mathcal{L}^+ \rho) \ln \rho. \quad (4.9)$$

Using the invariance of the Shannon entropy under the conservative (Euler) part of (3.5a), we get

$$\begin{aligned} \frac{dS}{dt}(\rho) &= - \int d\omega (\mathcal{L}_d^+ \rho) \ln \rho \\ &= \sum_k \gamma_k \int d\omega X_k \frac{\partial \ln \rho}{\partial \omega_k} \\ &= \sum_k \gamma_k \int d\omega X_k \left(\frac{X_k}{\rho} - \beta_k \bar{\omega}_k \right) \\ &= \sum_k \gamma_k \left\langle \left[\frac{X_k}{\rho} \right]^2 \right\rangle - \nu \sum_k k^2 \int d\omega \bar{\omega}_k X_k. \end{aligned} \quad (4.10)$$

Minus the second term, it reads

$$\begin{aligned} \nu \sum_k k^2 \int d\omega \bar{\omega}_k X_k &= \nu \sum_k k^2 \int d\omega \bar{\omega}_k \left(\frac{\partial \rho}{\partial \omega_k} + \beta_k \omega_k \rho \right) \\ &= \nu \sum_k \beta_k k^2 \int d\omega \rho \left(|\omega_k|^2 - \frac{1}{\beta_k} \right). \end{aligned} \quad (4.11)$$

Substituting (4.11) into (4.10) and then (4.10) into (4.8), we immediately obtain the desired identity (4.6).

B. Enstrophy network

The situation can now be summarized as follows. Locally, in the stationary measure, we have

$$\sum_\ell \langle J_{\ell k} \rangle = \langle J_k \rangle = \langle \bar{\omega}_k F_k(\omega) \rangle$$

and globally,

$$\sum_k \langle J_k \rangle = 0. \quad (4.12)$$

For the enstrophy dissipation,

$$\langle S \rangle = \sum_k \beta_k \langle J_k \rangle > 0. \quad (4.13)$$

We have here formally the same situation as for a heat conduction network, as considered, e.g., in Ref. [24]. The relations (4.12) and (4.13) do not, of course, uniquely determine the mean enstrophy currents, but their direction or sign is thermodynamically determined by analogy with heat conduction.

Let us first consider the typical case in which the strengths γ_k are nonzero only for a neighborhood of $k=0$, say $\gamma_k=1$ when $|k| \leq \eta$, and outside that large wavelength regime: $\gamma_k \downarrow 0, |k| > \eta$. In terms of heat conduction, it would mean that the temperatures $T_k = 1/(\nu k^2)$ are decreasing outward in the disk for $|k| \leq \eta$ and fall to $T_k=0$ outside ($|k| > \eta$). Clearly then, there will be a heat current toward increasing $|k|$ or, here, an enstrophy current towards smaller wavelengths. In other words, the enstrophy current is a kind of nonlocal heat current the direction of which is determined by the Second Law. Because of the nonlocality of the term $F_k(\omega)$, the current will not stop at the boundary of the disk but will be more and more suppressed when regarding $J_{k\ell}$ for k inside and ℓ outside the disk. For really large ℓ there is no longer a visible local heat current. That seems compatible with the observations [8] that the enstrophy cascade remains pretty localized around the forcing window.

In general, however, when all $\gamma_k > 0$ are active, we have a ‘‘temperature’’ profile $\gamma_k/(\nu k^2)$ that can, of course, be complicated. If the γ_k depend only on $|k|$, we have in essence a one-dimensional heat conduction problem (along the radial direction).

C. Fluctuations

Looking back at (4.7) and (4.8), we found the mean enstrophy as the change of entropy in the environment S plus the change of (Shannon) entropy due to the stochastic dynamics in the system. Its stationary mean ($\langle R \rangle > 0$) is strictly positive. We will now look at its fluctuations. More precisely, we consider the R of (4.4) and ask for its probability distribution. Since by construction (4.2)

$$P_\mu^\tau(\xi) = e^{R(\xi)} P_{\pi\mu}^\tau(\Theta \xi),$$

we have that

$$\int f(\Theta\xi)dP_{\pi\mu}^{\tau} = \int f(\xi)e^{-R(\xi)}dP_{\mu}^{\tau}(\xi) \quad (4.14)$$

is exactly valid for all observations f and for all times τ . The relation (4.14) is called a fluctuation symmetry, (see, e.g., Ref. [25]), because it generates the so-called fluctuation theorem for the entropy production as first formulated in Refs. [26–28]. Remember that R equals the S up to a temporal boundary term [see (4.4) and (4.5)].

One of the consequences of the fluctuation symmetry (4.14) is that

$$\frac{\text{Prob}[R < 0]}{\text{Prob}[R > 0]} = \langle e^{-R} | R > 0 \rangle, \quad (4.15)$$

which is sometimes easier to check numerically and experimentally. Roughly speaking, this last relation tells us that the probability of observing the inverse cascade for the enstrophy is exponentially smaller than the probability of observing the direct cascade.

V. CONCLUSIONS

The main conclusion is derived from the analogy with a heat conduction network. Above and beyond all detailed physical mechanisms that give rise to the direct enstrophy cascade in two-dimensional turbulence stands the Second Law of Thermodynamics for the entropy (4.1), which gives a direction to the enstrophy flow. The relevant parameter is the ratio $\gamma_k/(\nu k^2)$, which plays the role of an effective tempera-

ture of an enstrophy reservoir to which each mode k is coupled. If the forcing is restricted to a finite window, then the temperature outside is effectively equal to zero. The conservative part in the enstrophy conduction is nonlocal but does not contribute to the dissipation.

We have identified a general entropy function [(4.1) and (4.4)], also in the transient regime [see (4.7) and (4.8)]. We have shown that the stationary entropy production is strictly positive. It provides the general mechanism driving the direct enstrophy cascade. The fluctuations in the entropy satisfy the symmetry (4.14), which gives an estimate (4.15) of the relative probabilities of direct versus inverse cascades.

An important open question remains however. The above analogy is silent about the inverse energy cascade. We have not found a heat conduction analog that would reveal the inverse cascade for the energy dissipation in two-dimensional turbulence. Of course, as energy and enstrophy are entangled and spectrally related, the direct enstrophy cascade has direct consequences in the form of the inverse energy cascade. That point follows from the standard treatments (see also Sec. II D), as in Ref. [1], but has not been clarified in the present paper. In fact, a naive extension of the present formalism but for the energy would find the inverse cascade quite surprising as it seems to reduce entropy. We have not investigated whether the combination of dissipative currents, enstrophy, and energy, would still lead to a total positive entropy production, as expected thermodynamically. Clarifying that entropy balance remains one of the most intriguing problems of two-dimensional turbulence.

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