EFT in QFT context

We have seen that in the EFT approach a physical system is characterized by:

1. The set of relevant degrees of freedom
2. The symmetry of the system
3. An (a set of) expansion parameter(s)

In a QFT these 3 points become:

1. The set of fields (and their transf. properties under the symmetry of the system)
2. The symmetry, that will determine the allowed interactions
3. The power counting (a set of rules allowing to organize the Lagrangian as a series expansion)

The description should allowed predictions at any given precision in terms of a finite number of parameters, that can be determined by observation.

In Most of our examples, the power counting will involve a scale Λ. We will be interested in describing the system at typical energies $E \ll \Lambda$ with the idea that we do not need to know in detail the dynamics at $E \approx \Lambda$ or $E > \Lambda$.

In this case our expansion parameter will be something like $\frac{E}{\Lambda}$ or $\frac{m}{\Lambda}$ with $m \ll \Lambda$.

[Remark this is not always the case. For instance in nonrelativistic QED the expansion parameter is $\left(\frac{v}{c}\right)$, not $\left(\frac{E}{m_e}\right)$.]

In the TD approach we have a high-energy or "full" theory that in general will include both heavy $m_h \gtrsim \Lambda$ and light $m_\ell \ll \Lambda$ states (described by fields φ_h and φ_ℓ) described by a Lagrangian $\underbrace{\mathcal{L}_{UV}}_{\text{ultraviolet}}(\varphi_\ell, \varphi_h, \underbrace{C}_{\text{coupling constants}})$

and our task is to derive the low-energy Lagrangian $\mathcal{L}_{IR}(\varphi_\ell, C')$.

We will see that this operation is possible. Among other steps it involves:

1. Removal of the fields φ_h from $\mathcal{L}_{IR}$: we "integrate out" heavy particles

2. Full and low-energy theories should matched when tested at energies $E \ll \Lambda$.

3. The low-energy theory should be organized as a series expansion in $1/\Lambda$

$$\mathcal{L}_{IR}(\varphi_\ell, C') = \sum_n \mathcal{L}_{IR}^{(n)}$$

$\hookrightarrow$ includes appropriate power of $(1/\Lambda)$.

In the bottom-up approach the high-energy theory is unknown (or matching is too difficult to carry out, like for instance in QCD)

Typically we have some information: Lorentz, gauge invariance but not the details

We write down the most general $\mathcal{L}_{IR}(\varphi_e, c')$ consistent with the symmetry and the d.o.f.

$$\mathcal{L}_{IR}(\varphi_e, c') = \sum_n \mathcal{L}_{IR}^{(n)}$$

The required precision tell us where to stop the expansions. Parameters are fixed from observation.

Relevant, Irrelevant, Marginal operators

Assuming that we have a set of fields φ_ℓ describing light d.o.f., a symmetry and some cut-off scale Λ (like M_W for the Fermi theory) which operators $O_i(\varphi)$ should we include in a relativistic EFT?

Example (can be easily generalized)

1. 1 real scalar field $\varphi \equiv \varphi_\ell$
2. Symmetry: Lorentz and Z_2: $\varphi \to -\varphi$
3. cut-off Λ (can originate from a fundamental theory containing heavy d.o.f. φ_h, $m_h \simeq \Lambda$)

Work in d space time dimension
Most general Lagrangian:

$$\mathcal{L}_{EFT} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 - \sum_{n=1}^{\infty} c_n \varphi^{4+2n} - \sum_{n=1}^{\infty} d_n (\partial\varphi)^2 \varphi^{2n} + \dots$$

We would like to understand which of these terms is more important at low energies $E \ll \Lambda$
As we shall see the relative importance is dictated by a <u>dimensional analisis</u>

Exercise: determine the mass dimensions of all quantities using $[\hbar] = [c] = 0$
$[S] = 0$ $[d^d x] = -d$ $[\mathcal{L}_{EFT}] = +d$
$[\partial] = +1$

$[\varphi] = \frac{d-2}{2}$

	d	4
$[m]$	$+1$	$+1$
$[\lambda]$	$4-d$	0
$[c_n]$	$(4+2n) - d(n+1)$	$-2n$
$[d_n]$	$2n - dn$	$-2n$
$[\varphi^{4+2n}]$	$d - [c_n]$	$4+2n$
$[(\partial\varphi)^2 \varphi^{2n}]$	$d - [d_n]$	$4+2n$
$[\partial^{2q} \varphi^{2p}]$	$2q + p(d-2)$	$2(q+p)$

We can define dimensionless coefficients $\hat{\lambda}, \hat{c}_n, \hat{d}_n$ by rescaling

$$\lambda = \frac{\hat{\lambda}}{\Lambda^{d-4}} \qquad c_n = \frac{\hat{c}_n}{\Lambda^{d(n+1)-(4+2n)}} \qquad d_n = \frac{\hat{d}_n}{\Lambda^{dn-2n}}$$

The Lagrangian reads:

$$\boxed{\mathcal{L}_{EFT} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2} m^2 \varphi^2 + \sum_i \frac{\hat{a}_i \, O_i}{\Lambda^{[O_i]-d}}}$$

To study the relative importance of the different terms in the classical action we can rescale the space time coordinates

$$x^\mu = e^{+\alpha} x'^\mu$$

With x' fixed:

$$\begin{cases} \alpha \to -\infty & \text{small distance} \quad (\text{large } E) \\ \alpha \to +\infty & \text{large distance} \quad (\text{small } E) \end{cases}$$

At the same time, to maintain a canonical kinetic term we rescale the field:

$$\begin{cases} x^\mu = e^\alpha x'^\mu \\ \varphi(x) = e^{-[\varphi]\alpha} \varphi(x') = e^{\frac{2-d}{2}\alpha} \varphi(x') \end{cases}$$

The combination of the two transformation is called a <u>scale transformation.</u>
The reason why we want to keep invariant the kinetic term is that in the relativistic regime it dominates the path integral.

Exercise: show that the K.T. is invariant under a scale transformation

$$\int d^dx \, \frac{1}{2}(\partial\varphi)^2 = \int d^dx' \, \underbrace{e^{d\alpha} \, e^{-2\alpha} \, e^{(2-d)\alpha}}_{1} \, \frac{1}{2}(\partial_{x'}\varphi(x'))$$

Exercise: show that the result of a scale transformation in $\mathcal{L}_{EFT}$ is

$$\boxed{\mathcal{L}_{EFT} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2} m^2 e^{2\alpha} \varphi^2 + \sum_i \frac{\hat{a}_i}{\Lambda^{[O_i]-d}} e^{(d-[O_i])\alpha} O_i}$$

Under a scale transformation:

$$\boxed{O_i \to e^{(d-[O_i])\alpha}\, O_i}$$

Hence the classification:

$d > [O_i]$ relevant operator
$d = [O_i]$ marginal operator
$d < [O_i]$ irrelevant operator

For $\alpha \to \infty$ (smaller E) relevant ones are more important, marginal are insensitive to E and irrelevant are less important.

① Example of irrelevant are the Fermi ET and its corrections, suppressed by additional powers of $\left(\frac{E^2}{M_W^2}\right)$: $d = 4$ $[\psi\psi\psi\psi] = 6$ $d - [O] = -2$

Remark: leading "irrelevant" operators can be very important, when they describe effects forbidden by relevant and marginal operators. For instance:

- β - decay } in Fermi → forbidden in
  μ - decay } theory → QED + QCD
- Neutrino masses } → forbidden in the SM
- proton decay }
- violation of P and CP in Fermi theory → P and CP conserved in QED + QCD

light-by-light scattering $\gamma\gamma \to \gamma\gamma$ → not expected at LO in QED

(2) relevant operators like $m^2\varphi^2$ in $d=4$ or $m\bar{\Psi}\Psi$ in $d=4$ are part of the EFT, but they set up an infrared cutoff:
if we produce φ or Ψ particles in our experiment, we should have energies $E > m$.
But we expect the EFT to break down near $E \approx \Lambda$. We have a window of validity:

if m is to close to Λ this window shrinks and the EFT become useless.
E.g. imagine the Fermi theory describing the μ-decay with $m_\mu \approx m_W$. In this regime we need the full e.w. theory.
We can say that

irrelevant operators	set	Λ_{UV}
relevant operators	set	Λ_{IR}

(3) Other theories can have a different power counting. For instance in non-relativistic theories t and $\vec{x}$ can have a different scaling. Also relativistic theories with 1 heavy component (like atoms in the Rayleigh scattering).

Exercises

(1) Consider the mass operator for scalar φ, Dirac fermion Ψ and vector boson A_μ fields in d dimensions:

$$m^2 \varphi^2 \qquad m\bar{\Psi}\Psi \qquad m^2 A_\mu A^\mu$$

Can the operators be marginal in some dimension d?

We have

$[\varphi] = [A_\mu] = \frac{d-2}{2}$ and $[\Psi] = \frac{d-1}{2}$

$[\varphi^2] = [A_\mu A^\mu] = d-2 < d$ always

$[\bar{\Psi}\Psi] = d-1 < d$ always

$\rightarrow$ mass operators are relevant in any dimension d.

(2) Find d such that $(\bar{\Psi}\Psi)^2$ is marginal

$[(\bar{\Psi}\Psi)^2] = 2d-2 = d \qquad d=2$

(3) Find d such that φ^6 is marginal

$[\varphi^6] = 3d-6 = d \qquad d=3$

(4) Classify the operator $\varphi\Box\varphi^2$ for integer d

$[\varphi\Box\varphi^2] = 3\,\frac{d-2}{2} + 2 = \frac{3}{2}d - 1$

d	$[\varphi\Box\varphi^2]$	
1	$1/2 < 1$	relevant
2	2	marginal
3	$7/2 > 3$	irrelevant
4	$5 > 4$	irrelevant
⋮	⋮	⋮