BIREFRINGENCE OF VACUUM

Due to the non-linear terms induced at low-energy the vacuum behaves as a birefringent medium, i.e. photons have different propagation velocities depending on their polarization.
[Birefringence: optical property of a material having a refractive index that depends on the polarization and propagation direction of light]
common in materials like e.g. calcite.

① Equations of motion from $\mathcal{L}_{IR}$

Define $\mathcal{F} \equiv \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2}(E^2 - B^2)$ $\qquad$ $\mathcal{G} \equiv \frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} = -\vec{E}\cdot\vec{B}$

$$\boxed{\mathcal{L}_{IR} = -\mathcal{F} + c_1 \mathcal{F}^2 + c_2 \mathcal{G}^2}$$

We have seen that
$$\begin{cases} c_1 = \frac{\alpha^2}{m^4}\left(a_1 + \frac{a_2}{2}\right) \\ c_2 = \frac{\alpha^2}{4m^4} a_2 \end{cases}$$

$$\frac{\partial \mathcal{F}}{\partial \partial_\alpha A_\beta} = \frac{1}{2} F_{\mu\nu} \frac{\partial}{\partial \partial_\alpha A_\beta}(2\partial^\mu A^\nu) = F^{\alpha\beta}$$

$$\frac{\partial \mathcal{G}}{\partial \partial_\alpha A_\beta} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \frac{\partial}{\partial \partial_\alpha A_\beta} F_{\mu\nu} F_{\rho\sigma} \times \frac{1}{4}$$

$$= 2\times\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} [(g_{\mu\alpha} g_{\nu\beta} - \cancel{g_{\mu\beta} g_{\nu\alpha}}) F_{\rho\sigma} + F_{\mu\nu}(g_{\rho\alpha} g_{\sigma\beta} - \cancel{g_{\rho\beta} g_{\sigma\alpha}})] \times 1/4$$

$$= (\epsilon^{\alpha\beta\rho\sigma} F_{\rho\sigma} + \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu}) \times 1/4$$

$$= \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} \equiv \tilde{F}^{\alpha\beta}$$

$$\boxed{\frac{\partial \mathcal{F}}{\partial \partial_\alpha A_\beta} = F^{\alpha\beta}} \qquad \boxed{\frac{\partial \mathcal{G}}{\partial \partial_\alpha A_\beta} = \tilde{F}^{\alpha\beta}}$$

$$\frac{\partial}{\partial \partial_\alpha A_\beta} \mathcal{L}_{1R} = - F^{\alpha\beta} + 2c_1 \mathcal{F} F^{\alpha\beta} + 2c_2 \mathcal{G} \tilde{F}^{\alpha\beta}$$

$$\partial_\alpha \frac{\partial}{\partial \partial_\alpha A_\beta} \mathcal{L}_{1R} = - \partial_\alpha F^{\alpha\beta} + 2c_1 \partial_\alpha(\mathcal{F} F^{\alpha\beta}) + 2c_2 \partial_\alpha(\mathcal{G} \tilde{F}^{\alpha\beta})$$

$$\boxed{\begin{aligned} &- \partial_\alpha F^{\alpha\beta} + 2c_1 \mathcal{F} \partial_\alpha F^{\alpha\beta} + 2c_2 \mathcal{G} \overbrace{\partial_\alpha \tilde{F}^{\alpha\beta}}^{\equiv 0 \ \text{Bianchi identity}} \\ &+ 2c_1 (\partial_\alpha \mathcal{F}) F^{\alpha\beta} + 2c_2 (\partial_\alpha \mathcal{G}) \tilde{F}^{\alpha\beta} = 0 \end{aligned}}$$

classical, nonlinear eom.

We now linearize these equation by assuming

$$A_\mu = \underbrace{A_\mu^{cl}}_{\text{classical background field}} + \overbrace{A'_\mu}^{\text{fluctuation}}$$

fixed by experimental setting

We keep only linear terms in A'_μ (omitting the $'$)

Moreover we assume <u>constant</u> E^{cl} and B^{cl} fields:

$$\partial_\alpha (F_{\mu\nu}^{cl}) = 0$$

$$\partial_\alpha F^{\alpha\beta} \to \partial_\alpha F^{\alpha\beta}$$

$$\mathcal{F} \partial_\alpha F^{\alpha\beta} \to \mathcal{F}_{cl} \partial_\alpha F^{\alpha\beta}$$

$$(\partial_\alpha \mathcal{F}) F^{\alpha\beta} = \frac{1}{2} F_{\mu\nu} (\partial_\alpha F^{\mu\nu}) F^{\alpha\beta}$$

$$= \frac{1}{2} F^{\mu\nu}_{cl} F^{\alpha\beta}_{cl} \partial_\alpha F_{\mu\nu}$$

$$(\partial_\alpha G)\,\tilde F^{\alpha\beta} = \frac{1}{4}\,\frac{1}{\cancel{\chi}}\,\epsilon_{\mu\nu\rho\sigma}\,\cancel{\chi}\,F^{\mu\nu}\cdot\partial_\alpha F^{\rho\sigma}\cdot\tilde F^{\alpha\beta}$$

$$= \frac{1}{2}\,\tilde F_{\rho\sigma}\,\tilde F^{\alpha\beta}\,\partial_\alpha F^{\rho\sigma}$$

$$= \frac{1}{2}\,\tilde F^{\mu\nu}_{cl}\,\tilde F^{\alpha\beta}_{cl}\,\partial_\alpha F_{\mu\nu}$$

e.o.m.:

$$(1 - 2c_1\,\mathcal{F}_{cl})\,\partial_\alpha F^{\alpha\beta} =$$

$$= (c_1\,F^{\mu\nu}_{cl}F^{\alpha\beta}_{cl} + c_2\,\tilde F^{\mu\nu}_{cl}\tilde F^{\alpha\beta}_{cl})\,\partial_\alpha F_{\mu\nu}$$

$$(1 - 2c_1\,\mathcal{F}_{cl})(\Box g^{\alpha\beta} - \partial^\alpha\partial^\beta)\,A_\alpha$$

$$= 2\,(c_1\,F^{\mu\nu}_{cl}F^{\alpha\beta}_{cl} + c_2\,\tilde F^{\mu\nu}_{cl}\tilde F^{\alpha\beta}_{cl})\,\partial_\alpha\partial_\mu A_\nu$$

$$\boxed{\begin{aligned}&(1 - 2c_1\,\mathcal{F}_{cl})(\Box g^{\mu\nu} - \partial^\mu\partial_\nu)\,A_\nu\\ &= 2\,(c_1\,F^{\alpha\mu}_{cl}F^{\beta\nu}_{cl} + c_2\,\tilde F^{\alpha\mu}_{cl}\tilde F^{\beta\nu}_{cl})\,\partial_\alpha\partial_\beta A_\nu\end{aligned}}$$

linearized eom around classical static homogeneous E_{cl} B_{cl} fields.
In momentum space:

$$(1 - 2c_1\,\mathcal{F}_{cl})(k^2 g^{\mu\nu} - k^\mu k^\nu)\,A_\nu$$

$$= (2c_1\,b^\mu b^\nu + 2c_2\,\tilde b^\mu\tilde b^\nu)\,A_\nu$$

$$b^\mu \equiv F^{\alpha\mu} k_\alpha \qquad \tilde{b}^\mu \equiv \tilde{F}^{\alpha\mu} k_\alpha$$

Dispersion relation for Maxwell equations ($c_{1,2}=0$)
In Lorentz gauge $k^\nu A_\nu = 0$

$$k^2 A^\mu = 0 \quad \leftrightarrow \quad k^2 = 0$$

$$k^\mu = (\omega, \vec{k}) \qquad \omega^2 = |\vec{k}|^2 \qquad \omega = |\vec{k}|$$

Dispersion relations for modified equations in Lorentz gauge

$$\boxed{(1 - c_1 \mathcal{F}_{cl}) k^2 A^\mu = (2c_1 b^\mu b^\nu + 2c_2 \tilde{b}^\mu \tilde{b}^\nu) A_\nu}$$

Approximations:
We have 2 parameters, the classical field intensities $\epsilon \simeq |E_{cl}|, |B_{cl}|$ and the energy of the fluctuation $A_\mu \simeq \omega$
As we will see

$$k^2_{Maxwell} = 0 \quad \longrightarrow \quad k^2 \simeq \omega^2 \epsilon^2$$

So $\mathcal{F}_{cl} k^2 \simeq \epsilon^4 \omega^2$
Instead $b \sim \tilde{b} \sim \epsilon\omega \qquad b^2 \sim \tilde{b}^2 \sim \epsilon^2\omega^2$
For small field intensities we can drop the $\mathcal{F}_{cl}$ term in the left-hand side:

$$(k^2 g^{\mu\nu} - 2c_1 b^\mu b^\nu - 2c_2 \tilde{b}^\mu \tilde{b}^\nu) A_\nu = 0$$

According to our assumption $k^2 \approx \omega^2 \epsilon^2 \ll 1$

$$b^\mu \tilde{b}_\mu = F^{\alpha\mu} \tilde{F}^\beta{}_\mu \, k_\alpha k_\beta$$

$$(F^{\alpha\mu} \tilde{F}^\beta{}_\mu) = (\vec{E}\cdot\vec{B})\, g^{\alpha\beta}$$ (explicitly proved)

$$b^\mu \tilde{b}_\mu = (\vec{E}\cdot\vec{B})\, k^2 \approx \omega^2 \epsilon^4$$

$$b^\mu b_\mu = -\omega^2 \vec{E}^2 + (\vec{E}\cdot\vec{k})^2 - |\vec{k}|^2 \vec{B}^2 + (\vec{B}\cdot\vec{k})^2 + 2\omega\, \vec{k}\cdot(\vec{E}\times\vec{B})$$

$$\tilde{b}^\mu \tilde{b}_\mu = -\omega^2 \vec{B}^2 + (\vec{B}\cdot\vec{k})^2 - |\vec{k}|^2 \vec{E}^2 + (E\cdot\vec{k})^2 - 2\omega\, \vec{k}\cdot(\vec{B}\times\vec{E})$$

using $k^\mu \equiv (\omega, \vec{k})$

$$b^\mu b_\mu - \tilde{b}^\mu \tilde{b}_\mu = (|\vec{k}|^2 - \omega^2)(\vec{E}^2 - \vec{B}^2) \approx \omega^2 \epsilon^4$$

Neglecting terms of $O(\omega^2 \epsilon^4)$ we have :

$$\boxed{b\,\tilde{b} = 0 \qquad b^2 = \tilde{b}^2}$$

Moreover, neglecting small terms $O(\omega^2 \epsilon^4)$ we have

$$b^2 = \tilde{b}^2 = -(\vec{k}\times\vec{E})^2 - (\vec{k}\times\vec{B})^2 + 2\,|\vec{k}|\,\vec{k}\cdot(\vec{E}\times\vec{B})$$

$$\boxed{b^2 = \tilde{b}^2 = -\omega^2 Q^2 \leq 0} \qquad Q^2 \approx \epsilon^2$$

here $\omega = |\vec{k}|$ is assumed.

$(\vec{k}\times\vec{B})^2 \equiv |\vec{k}|^2 \vec{B}^2 - (\vec{B}\cdot\vec{k})^2$ used

In fact only $2|\vec{k}|\,\vec{k}\cdot(\vec{E}\times\vec{B})$ can be positive and it is maximized when $(\vec{k},\vec{E},\vec{B})$ form an orthogonal set, giving $2|\vec{k}|^2 EB$. In this case $b^2=\tilde{b}^2 = -|\vec{k}|^2(|E|-|B|)^2 \leq 0$

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The e.o.m.

$$(k^2 g^{\mu\nu} - 2c_1 b^\mu b^\nu - 2c_2 \tilde{b}^\mu \tilde{b}^\nu)A_\nu = 0$$

has two independent solutions given by

$$A_\nu = \beta b_\nu + \tilde{\beta}\tilde{b}_\nu$$

$$(k^2 g^{\mu\nu} - 2c_1 b^\mu b^\nu - 2c_2 \tilde{b}^\mu \tilde{b}^\nu)b_\nu =$$

$$k^2 b^\mu + 2c_1\omega^2 Q^2 b^\mu = 0 \qquad \text{upto } \epsilon^4\omega^2 \text{ terms}$$

$$k^2 + 2c_1\omega^2 Q^2 = 0 \quad \leftrightarrow b_\nu$$

similarly

$$k^2 + 2c_2\omega^2 Q^2 = 0 \quad \leftrightarrow \tilde{b}_\nu$$

Defining a refractive index n: $k^\mu \equiv (\omega, \overset{\omega}{\vee} n\vec{e})$ we have:

$$\omega^2(1+2c_{1,2}Q^2) - \omega^2 n^2 = 0$$

$$n^2 = 1 + 2c_{1,2}Q^2$$

$$\boxed{n_{1,2} \simeq 1 + c_{1,2}Q^2}$$

In terms of velocities, we have

$$\cancel{\omega}\Delta t = n_{1,2}\cancel{\omega}\Delta x$$

$$\rightarrow v = \frac{\Delta x}{\Delta t} = \frac{1}{n_{1,2}} \approx (1 - c_{1,2}Q^2)$$

Notice: to avoid superluminous signals we need $v < 1$ $\rightarrow$ $c_{1,2} > 0$

$\ast$

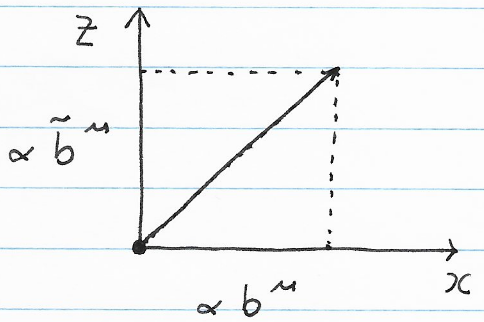
A typical experimental set-up exploits a constant magnetic field $\vec{B} = (0, 0, B)$ and a light ray propagating in an orthogonal direction $\vec{n} = (0, 1, 0)$ $\leftrightarrow$ $k^\mu = (\omega, 0, \omega n, 0)$

In this case:

$$b^\mu = (0, \omega n B, 0) \qquad \tilde{b}^\mu = (0, 0, 0, \omega B)$$

One chooses a linearly polarized beam.

[y down going]



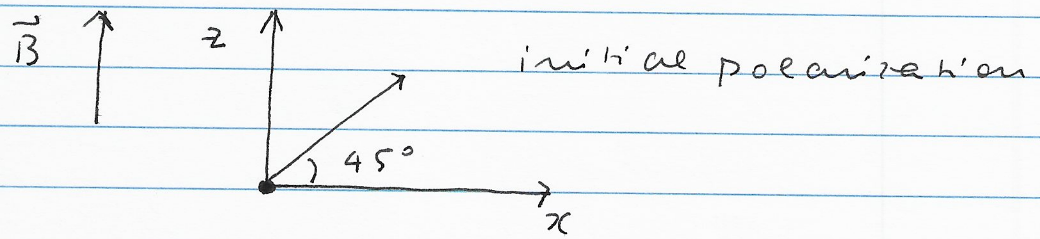
Since the 2 components have a different propagation velocity, after travelling a length L the phases experience a shift $\Delta\varphi$

$$\Delta\varphi = \omega n_2 L - \omega n_1 L$$
$$\approx \frac{2\pi}{\lambda} \Delta n \, L$$

$$\Delta n = (c_2 - c_1) Q^2 = (c_2 - c_1) B^2$$

$$\Delta\varphi \approx \frac{2\pi}{\lambda} (c_2 - c_1) B^2 L$$

To understand the orientation of A'_ν (polarization)
We consider a region with constant magnetic field $\vec{B}$ along the z-axis, and an e.m. field A'_ν



propagating in a direction perpendicular to $\vec{B}$ (for instance along the y axis).

$\vec{B} = (0, 0, B)$ $\qquad\qquad$ $\vec{e} = (0, 1, 0)$

$$A'_\nu = \alpha\, b_\nu + \tilde{\alpha}\, \tilde{b}_\nu$$

$b_\nu = k^\alpha F^{ce}_{\alpha\nu}$ $\qquad\qquad$ $\tilde{b}_\nu = k^\alpha \tilde{F}^{ce}_{\alpha\nu}$

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

$(\omega, n_1 \omega \vec{e})$ $\qquad\qquad$ $(\omega, n_2 \omega \vec{e})$

$= (\omega, 0, n_1\omega, 0)$ $\qquad\qquad$ $= (\omega, 0, n_2\omega, 0)$

$$b_\nu = \omega\left(\cancel{F^{ce}_{0\nu}} + n_1 F^{ce}_{2\nu}\right) = \omega n_1 (0, B, 0, 0)$$

$$\tilde{b}_\nu = \omega\left(\tilde{F}^{ce}_{0\nu} + \cancel{n_2 \tilde{F}^{ce}_{2\nu}}\right) = \omega (0, 0, 0, B)$$

$$F^{\mu\nu}_{ce} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -B_3 & 0 \\ 0 & B_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\cancel{\tilde{F}^{\mu\nu}_{ce} = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & E_3 & 0 \end{matrix}}$$

$$\tilde{F}^{\mu\nu}_{ce} = \begin{pmatrix} 0 & 0 & 0 & -B_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B_3 & 0 & 0 & 0 \end{pmatrix}$$

b_ν is along x
$\tilde{b}_\nu$ is along z

Assume a monochromatic beam entering the B region (in the y direction) with linear polarization

$$A'_\nu(t=0) = (0, A, 0, A)$$

$$A = \bar{A} \sin \cancel{} (ky - \omega t)\Big|_{t=0}$$

The x and z component ($\sim b_\nu$ and $\tilde{b}_\nu$ respectively) propagate with different velocities and after a length L they will accumulate a phase difference $\Delta\varphi$

$$\Delta\varphi = (k_2 - k_1) L$$
$$= \omega (n_2 - n_1) L$$
$$= \frac{2\pi}{\lambda} \Delta n L$$

$$\Delta n = (C_2 - C_1) Q^2 = (C_2 - C_1) B^2$$

$$\Delta\varphi = \frac{2\pi}{\lambda} (C_2 - C_1) B^2 L$$

$$C_1 = \frac{\alpha^2}{m^4} \frac{8}{45} \qquad C_2 = \frac{\alpha^2}{m^4} \frac{1}{2} \frac{7}{45}$$

$$|C_1 - C_2| = \frac{\alpha^2}{m^4} \frac{16-7}{90} = \frac{\alpha^2}{m^4} \cdot \frac{1}{10}$$

$$A'_v(L) = \bar{A}\,(0,\ \sin(k_1 L - \omega t),\ 0,\ \sin(k_2 L - \omega t))$$

$$\begin{cases} A_1'(L) = \bar{A} \sin(k_1 L - \omega t) \\ A_3'(L) = \bar{A} \sin(k_1 L - \omega t + \Delta\varphi) \end{cases}$$

describing an ellipse

$$x(t) = \bar{x} \sin(\varphi_x - \omega t)$$

$$z(t) = \bar{x} \sin(\varphi_z - \omega t) =$$

$$= \bar{x} \sin(\varphi_x - \omega t + \Delta\varphi)$$

$$z(t) = \bar{x} \sin(\varphi_x - \omega t) \cos\Delta\varphi + \bar{x}\cos(\varphi_x - \omega t) \sin\Delta\varphi$$

$$z(t) - x(t)\cos\Delta\varphi = \bar{x}\cos(\varphi_x - \omega t)\sin\Delta\varphi$$

$$(z(t) - x(t)\cos\Delta\varphi)^2 = \sin^2\Delta\varphi\ \bar{x}^2 (1 - \sin^2(\varphi_x - \omega t))$$

$$= \sin^2\Delta\varphi\ \bar{x}^2 - \sin^2\Delta\varphi\ x^2(t)$$

$$= z^2(t) + x^2(t)\cos^2\Delta\varphi - 2x(t)z(t)\cos\Delta\varphi$$

$$+ x^2(t)\sin^2\Delta\varphi = \sin^2\Delta\varphi\ \bar{x}^2$$

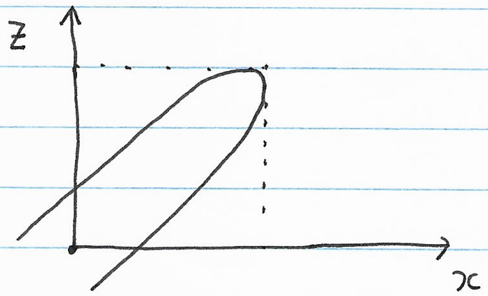
$$= x^2(t) + z^2(t) - 2(x(t)z(t))\cos\Delta\varphi = \sin^2\Delta\varphi\ \bar{x}^2$$

$\Delta\varphi = 0 \rightarrow (x(t) - z(t))^2 = 0$ o.k.

$\cos\Delta\varphi = 0 \rightarrow x^2(t) + z^2(t) = \bar{x}^2$ o.k.

$$\boxed{A_1'^2 + A_3'^2 - 2A_1'A_3'\cos\Delta\varphi = \bar{A}^2\sin^2\Delta\varphi}$$

and this modifies the linear into an elliptic polarization.



Some number:
sensitivity of PVLAS experiment : $\Delta\varphi \approx 10^{-7} \mathrm{rad}/\sqrt{Hz}$

$(C_2 - C_1)_{QED} \approx 4 \cdot 10^{-24}\ T^{-2}$

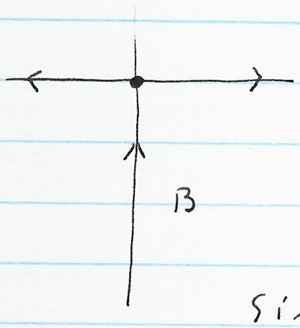
$B \approx 2.5\ T \quad L \approx 1m \quad \to (C_2 - C_1) B^2 L \approx 2.5\ 10^{-23}\ m$

$\lambda \approx 1 \mu m$

$\Delta\varphi \approx 10^{-17}\ \mathrm{rad} \xrightarrow[\text{cavity}]{\text{enhanced by}} 10^{-11}\ \mathrm{rad} \ll 10^{-7}\ \mathrm{rad}$

Vacuum birefringence tests

Astrophysical evidence from magnetars (Neutron stars with large magnetic field)
Linearly polarized light is expected to be emitted from the surface of a NS where there is point by point a magn. field



← linear polarization
light emitted perpendicularly to the sheet.

Since B changes on the SN surface small net effects are expected.
A net linear polarization can survive in the presence of a birefringent vacuum.
Lab experiments

→ PVLAS

→ BMV

Exercise

Expand the Born-Infeld Lagrangian (1930) up to $O(4)$ in the electromagnetic fields and find the correspondence with the Euler-Heisenberg Lagrangian

BI : non-linear electrodynamics motivated by the search for a maximum value of E, B to remove divergence in m_e in classical th.

$$\mathcal{L}_{BI} = -b^2 \sqrt{-\det\left(g_{\mu\nu} + \frac{F_{\mu\nu}}{b}\right)}$$

$[b] = 2$

$$-\det \begin{pmatrix} 1 & E^1 & E^2 & E^3 \\ -E^1 & -1 & -B^3 & B^2 \\ -E^2 & B^3 & -1 & -B^1 \\ -E^3 & -B^2 & +B^1 & -1 \end{pmatrix}$$

describes also a gauge field on a D-brane

$$= 1 + \frac{1}{2b^2} F_{\mu\nu} F^{\mu\nu} \blacksquare \frac{1}{16} \frac{(F\tilde{F})^2}{b^4}$$

$$= 1 + \frac{2\mathcal{F}}{b^2} - \frac{\mathcal{G}^2}{b^4}$$

$$\mathcal{L} = -b^2 \sqrt{1 + \frac{2}{b^2}\mathcal{F} - \frac{\mathcal{G}^2}{b^4}}$$

$$= -b^2 \left(1 + \frac{\cancel{2}}{2b^2}\mathcal{F} - \frac{\mathcal{G}^2}{2b^4} - \frac{\mathcal{F}^2}{2b^4} + \dots\right)$$

$$= -b^2 - \mathcal{F} + \frac{\mathcal{G}^2}{2b^2} + \frac{\mathcal{F}^2}{2b^2} + \dots \rightarrow a_1' = a_2'$$