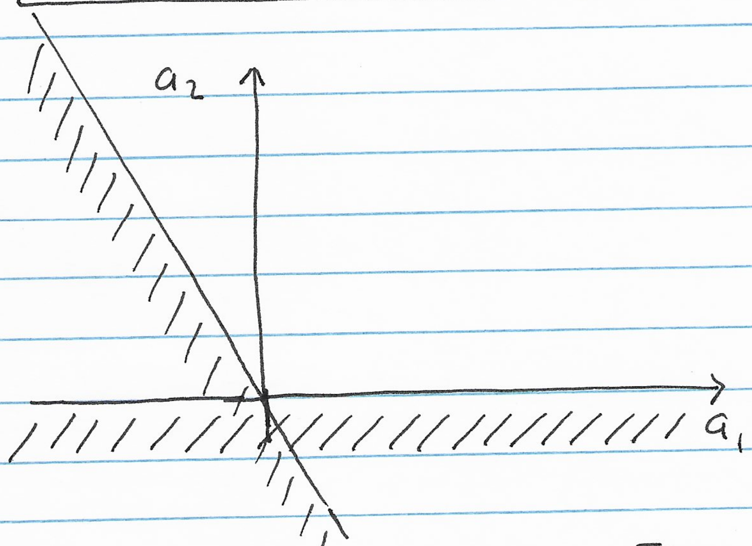
Forward scattering amplitude and bounds on the coefficients a_1, a_2.

In a bottom-up perspective, the coefficients a_1 and a_2 of $\mathcal{L}_{IR}$ are arbitrary real numbers. However it is possible to show that under general assumptions always satisfied by any $\mathcal{L}_{UV}$ (not only QED) they obey some positivity conditions, i.e. they cannot be choosen completely at random. As we will see:

$$\boxed{2a_1 + a_2 > 0 \qquad a_2 > 0} \leftrightarrow a_1', a_2' > 0$$



Landscape of possible theories. The hatched region is the "swampland": no UV sensible theory can give rise to (a_1, a_2) in this region.

Determining the swampland in the landscape of low-energy EFT whose UV completion is string theory is one of the most active field of research today.

To show that $2a_1 + a_2 > 0$ and $a_2 > 0$ we will exploit 3 general ingredients

1. Analyticity of scattering amplitudes (considered as functions of the Mandelstam variables s, t, u extended to the complex plane)

2. Crossing symmetry : a relation between
$A(k_1, \lambda_1 ; k_2 \lambda_2 \,|\, k_3 \lambda_3 ; k_4, \lambda_4)$
and $A(\underbrace{-k_3, -\lambda_3 ; k_2, \lambda_2 \,|\, -k_1}\, -\lambda_1 ; k_4, \lambda_4)$

1st and 3rd (k, λ) variables exchanged

3. optical theorem : relating for a 2 particle scattering

$$\mathrm{Im}\, A(i \to f)\Big|_{i=f} = 2 E_{CM} |\vec{p}| \sum_X \sigma(i \to X)$$

$\hookrightarrow$ momentum of 1 particle in CM frame

= massless particles

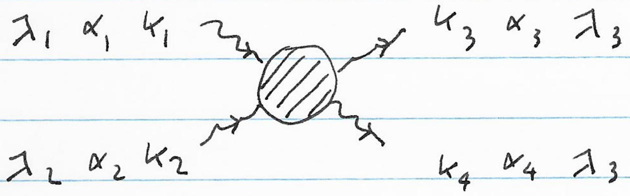
$$= s \sum_X \sigma(i \to X) = s\, \sigma_{tot}(i \to X)$$

$k_1 = (k, 0, 0, k)$ $\quad k_2 = (k, 0, 0, -k)$ $\quad s = (k_1 + k_2)^2 = 4k^2$

$$= 4k \cdot k$$

$$= 2 E_{CM} \cdot k$$

To prove the positivity relation we study the forward scattering amplitude of light-by-light scattering with fixed photon helicities



(Now $k_{3,4}$ are outcoming i.e $k_{3,4} \to -k_{3,4}$ in our previous Feynman rule)

λ_i are the photon helicities : ± 1

$$A_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} = \varepsilon^{*\alpha_4}_{\lambda_4}(k_4)\, \varepsilon^{*\alpha_3}_{\lambda_3}(k_3)\, \varepsilon^{\alpha_2}_{\lambda_2}(k_2)\, \varepsilon^{\alpha_1}_{\lambda_1}(k_1) \times$$
$$\times \mathcal{M}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(k_1, k_2, k_3, k_4)$$

$k_i^2 = 0$ for massless particles

we recall the Mandelstam variables

$s = (k_1 + k_2)^2$

$t = (k_1 - k_3)^2$

$u = (k_1 - k_4)^2$

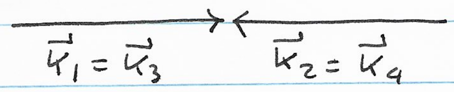
with $s + t + u = 0$ (massless kinematics)

Now consider the forward scattering:

$k_3 = k_1 \qquad k_4 = k_2$

For instance, in the CM frame

$k_1 = (k, 0, 0, k) \qquad k_2 = (k, 0, 0, -k) = k_4$
$= k_3$



In the CM frame the polarization vectors read:

$$\epsilon^{\alpha_1}_{\lambda_1}(k_1) = (0, -\lambda_1, -i, 0)/\sqrt{2}$$
$$\epsilon^{\alpha_2}_{\lambda_2}(k_2) = (0, +\lambda_2, -i, 0)/\sqrt{2}$$

$$\epsilon^{\alpha_3 *}_{\lambda_3}(k_1) = (0, -\lambda_3, i, 0)/\sqrt{2}$$
$$\epsilon^{\alpha_4 *}_{\lambda_4}(k_2) = (0, +\lambda_4, i, 0)/\sqrt{2}$$

Notice that $\boxed{\epsilon_\lambda(k) \cdot k' \equiv 0}$ whatever k and k' can be.

Moreover we have $t = 0$ $u = -s$

→ This means that we can parametrize the most general amplitude $\mathcal{M}_{\alpha_1\alpha_2\alpha_3\alpha_4}$ as

$$\boxed{\begin{aligned}\mathcal{M}_{\alpha_1\alpha_2\alpha_3\alpha_4} = {} & A(s)\, g_{\alpha_1\alpha_3} g_{\alpha_2\alpha_4} + B(s)\, g_{\alpha_1\alpha_4} g_{\alpha_2\alpha_3} \\ & + C(s)\, g_{\alpha_1\alpha_2} g_{\alpha_3\alpha_4} + \ldots\end{aligned}}$$

↑ terms proportional to some $(k_i)_\alpha$ that do not contribute to A

<u>Exercise</u>: The above is completely general. Compute $A(s)$, $B(s)$ and $C(s)$ in $\mathcal{L}_{IR}$

In units of $\left(i \frac{\alpha^2}{m_e^4}\right)$ we have:

$$\underbrace{(k_1 \cdot k_2)^2}_{\frac{s^2}{4}} \Big[\left(2a_1 + \frac{a_2}{2}\right)\left(g_{\alpha_1\alpha_2} g_{\alpha_3\alpha_4} + g_{\alpha_1\alpha_4} g_{\alpha_2\alpha_3}\right) + \cancel{\chi} a_2\, g_{\alpha_1\alpha_3} g_{\alpha_2\alpha_4}\Big] + \ldots$$

Crossing symmetry

$A_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(s, t, u)$ remains invariant under the crossing transformation

$k_1 \to -k_3$ helicity (1) → − helicity (3)
$k_3 \to -k_1$ helicity (3) → − helicity (1) (1)
variables 2, 4 unchanged

(1) Exchanges initial and final state reversing the sign of the kinematical variables

Assuming λ_i now denote helicities $\lambda_i = \pm 1$ we have

$$\boxed{A_{-\lambda_3 \lambda_2 -\lambda_1 \lambda_4}(u, t, s) = A_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(s, t, u)}$$

indeed $s = (k_1 + k_2)^2 \to (k_2 - k_3)^2 = (k_4 - k_1)^2 = u$
$t = (k_1 - k_3)^2 \to (k_1 - k_3)^2 = t$

For forward scattering $t = 0$ $u = -s$
$t = 0 \to t = 0$ $s \to u = -s$

$$\boxed{A_{-\lambda_3 \lambda_2 -\lambda_1 \lambda_4}(-s) = A_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(s)}$$

Remark
In general, if $A(s)$ is a physical amplitude the crossing transformation relates it to an unphysical amplitude $(-s < 0)$

<u>Exercise</u> Enforce crossing symmetry on

$$A_{\lambda_1\lambda_2\lambda_3\lambda_4}(s) = \epsilon^{*\alpha_4}_{\lambda_4}\,\epsilon^{*\alpha_3}_{\lambda_3}\,\epsilon^{\alpha_2}_{\lambda_2}\,\epsilon^{\alpha_1}_{\lambda_1}\times$$

$$\times\left[A(s)\,g_{\alpha_1\alpha_3}\,g_{\alpha_2\alpha_4} + B(s)\,g_{\alpha_1\alpha_4}\,g_{\alpha_2\alpha_3} + C(s)\,g_{\alpha_1\alpha_2}\,g_{\alpha_3\alpha_4}\right]$$

$$\begin{aligned}
= {} & A(s)\ \epsilon^*_{\lambda_3}(k_1)\cdot\epsilon_{\lambda_1}(k_1)\ \ \epsilon^*_{\lambda_4}(k_2)\cdot\epsilon_{\lambda_2}(k_2)\\
+ {} & B(s)\ \epsilon^*_{\lambda_4}(k_2)\cdot\epsilon_{\lambda_1}(k_1)\ \ \epsilon^*_{\lambda_3}(k_1)\,\epsilon_{\lambda_2}(k_2)\\
+ {} & C(s)\ \epsilon_{\lambda_2}(k_2)\cdot\epsilon_{\lambda_1}(k_1)\ \ \epsilon^*_{\lambda_4}(k_2)\,\epsilon^*_{\lambda_3}(k_1)
\end{aligned}$$

using
$$\begin{aligned}
\epsilon^{\alpha_1}_{\lambda_1}(k_1) &= \tfrac{1}{\sqrt{2}}\,(0, -\lambda_1, -i, 0)\\
\epsilon^{\alpha_2}_{\lambda_2}(k_2) &= \tfrac{1}{\sqrt{2}}\,(0, +\lambda_2, -i, 0)\\
\epsilon^{\alpha_3}_{\lambda_3}(k_1)^* &= \tfrac{1}{\sqrt{2}}\,(0, -\lambda_3, +i, 0)\\
\epsilon^{\alpha_4}_{\lambda_4}(k_2)^* &= \tfrac{1}{\sqrt{2}}\,(0, +\lambda_4, +i, 0)
\end{aligned}$$

$$A_{\lambda_1\lambda_2\lambda_3\lambda_4}(s) = \frac{1}{4}\Big[A(s)\,(1+\lambda_1\lambda_3)(1+\lambda_2\lambda_4) + B(s)\,(1-\lambda_1\lambda_4)(1-\lambda_2\lambda_3) + C(s)\,(1+\lambda_1\lambda_2)(1+\lambda_3\lambda_4)\Big]$$

- $A_{++++}(s) = A_{----}(s) = A(s) + C(s)$

  $A_{+++-}(s) = A_{---+}(s) = \emptyset$ etc...
- $A_{++--}(s) = A_{--++}(s) = B(s) + C(s)$
- $A_{+-+-}(s) = A_{-+-+}(s) = A(s) + B(s)$

  $A_{+--+}(s) = A_{-++-}(s) = \emptyset$ ($\neq 0$ in general kinematics with $t \neq 0$)

crossing symmetry:

(1) $A_{-+-+}(-s) = A_{++++}(s)$

$A(-s) + B(-s) = A(s) + C(s)$

(2) $A_{++--}(-s) = A_{++--}(s)$

$B(-s) + C(-s) = B(s) + C(s)$

(2) $B(-s) - C(s) = B(s) - C(-s)$

$A(-s) - A(s) = C(s) - B(-s) = C(-s) - B(s)$

$= A(s) - A(-s)$

$$\rightarrow \boxed{\begin{aligned} A(s) &= A(-s) \\ C(s) &= B(-s) \end{aligned}}$$

The independent helicity amplitudes are

$$\boxed{\begin{aligned} A_{++++}(s) &= A_{----}(s) = A(s) + B(-s) \\ A_{++--}(s) &= A_{--++}(s) = B(s) + B(-s) \\ A_{+-+-}(s) &= A_{-+-+}(s) = A(s) + B(s) \end{aligned}}$$

Recall that in our EFT we have

$$\mathcal{M}_{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{i\alpha^2}{m_e^4} \cdot \underbrace{(k_1 \cdot k_2)^2}_{\downarrow \atop \frac{s^2}{4}} \left[\left(2a_1 + \frac{a_2}{2}\right)\left(g_{\alpha_1\alpha_2} g_{\alpha_3\alpha_4} + g_{\alpha_1\alpha_4} g_{\alpha_2\alpha_3}\right) + a_2\, g_{\alpha_1\alpha_3} g_{\alpha_2\alpha_4} \right]$$

$\rightarrow A(s) \equiv a_2 \dfrac{s^2}{4}$ $\qquad$ $B(s) = C(-s) \equiv \dfrac{4a_1 + a_2}{8} s^2$

in units of $\left(\dfrac{i\alpha^2}{m_e^4}\right)$

basis of linearly polarized photons :

$$\epsilon_x(k_1) = \epsilon_x^*(k_1) = (0, 1, 0, 0)$$
$$\epsilon_y(k_1) = \epsilon_y^*(k_1) = (0, 0, 1, 0)$$

$$\epsilon_x(k_2) = \epsilon_x^*(k_2) = (0, -1, 0, 0)$$
$$\epsilon_y(k_2) = \epsilon_y^*(k_2) = (0, 0, +1, 0)$$

apply a rotation of π in (x, y) plane.

$$\rightarrow \quad \epsilon_{\lambda_3}^*(k_1)\,\epsilon_{\lambda_1}(k_1) \quad \epsilon_{\lambda_4}^*(k_2)\,\epsilon_{\lambda_2}(k_2) = \delta_{\lambda_1\lambda_3}\ \delta_{\lambda_2\lambda_4}$$

$$\epsilon_{\lambda_4}^*(k_2)\,\epsilon_{\lambda_1}(k_1) \quad \epsilon_{\lambda_3}^*(k_1)\,\epsilon_{\lambda_2}(k_2) = (+1)\,\delta_{\lambda_1\lambda_4}(+1)\,\delta_{\lambda_2\lambda_3}$$

$$\epsilon_{\lambda_4}^*(k_2)\,\epsilon_{\lambda_3}^*(k_1)\ \epsilon_{\lambda_2}(k_2)\,\epsilon_{\lambda_1}(k_1) = (+1)\,\delta_{\lambda_3\lambda_4}\ \delta_{\lambda_1\lambda_2}(+1)$$

$$\rightarrow \quad A_{\lambda_1\lambda_2\lambda_3\lambda_4} = A(s)\,\delta_{\lambda_1\lambda_3}\,\delta_{\lambda_2\lambda_4} + B(s)\,\delta_{\lambda_1\lambda_4}\,\delta_{\lambda_2\lambda_3} + B(-s)\,\delta_{\lambda_1\lambda_2}\,\delta_{\lambda_3\lambda_4}$$

$$\rightarrow \quad \boxed{\begin{aligned} A_{xxxx} &= A(s) + B(s) + B(-s) = A_{yyyy} \\ A_{xyxy} &= A(s) = A_{yxyx} \\ A_{xyyx} &= A_{yxyx} = B(s) \\ A_{xxyy} &= A_{yyxx} = B(-s) \end{aligned}}$$

relations :

$$A_{xxyy}(s) = A_{xyyx}(-s)$$
$$A_{xxxx}(s) = A_{xyxy}(s) + A_{xyyx}(s) + A_{xxyy}(s)$$

By inspecting the 1-loop amplitude $A(s) \xrightarrow{|s|\to\infty} \log s, \log^2 s$ and the circle at ∞ does not contribute. Define

$$A_{\lambda_1\lambda_2}(s) \equiv A_{\lambda_1\lambda_2\lambda_1\lambda_2}(s)$$ (same polarization for 1 and 3)

$$\frac{d^2A_{\lambda_1\lambda_2}}{ds^2}(0) = \frac{1}{\pi i}\oint_c \frac{ds}{s^3} A_{\lambda_1\lambda_2}(s)$$

$$= \frac{1}{\pi i}\Bigg[\int_{4m^2}^{+\infty} \frac{ds}{s^3}\left(A_{\lambda_1\lambda_2}(\sigma+i\varepsilon) - A_{\lambda_1\lambda_2}(\sigma-i\varepsilon)\right)$$

$$+ \int_{-\infty}^{-4m^2} \frac{ds}{s^3}\left(A_{\lambda_1\lambda_2}(\sigma+i\varepsilon) - A_{\lambda_1\lambda_2}(\sigma-i\varepsilon)\right)$$

Remember that, by the Feynman prescription, the physical amplitude is:

$$A_{\lambda_1\lambda_2}(\sigma) \equiv \lim_{\varepsilon\to 0^+} A_{\lambda_1\lambda_2}(\sigma+i\varepsilon)$$

$$= \frac{1}{\pi i}\Bigg[\int_{4m^2}^{+\infty} \frac{ds}{s^3} \underbrace{\left(A_{\lambda_1\lambda_2}(\sigma+i\varepsilon) - A_{\lambda_1\lambda_2}(\sigma-i\varepsilon)\right)}_{2i\,\mathrm{Im}\,A_{\lambda_1\lambda_2}(\sigma)}$$

$$- \int_{4m^2}^{+\infty} \frac{ds}{s^3}\Big(\underbrace{A_{\lambda_1\lambda_2}(-\sigma+i\varepsilon) - A_{\lambda_1\lambda_2}(-\sigma-i\varepsilon)}_{-2i\,\mathrm{Im}\,A_{\lambda_1\lambda_2}(-\sigma)}\Big)\Bigg]$$

The amplitude also satisfies (Schwartz reflection principle):

$$A_{\lambda_1\lambda_2}(s^*) = A(s)^*$$

$\rightarrow$ Re $A_{\lambda_1\lambda_2}(s)$ is continuous while Im $A_{\lambda_1\lambda_2}(s)$ is not

$$\left.\frac{d^2 A_{\lambda_1\lambda_2}}{ds^2}\right|_{s=0} = \frac{2}{\pi}\int_{4m^2}^{+\infty}\frac{ds}{s^3}\left[\operatorname{Im} A_{\lambda_1\lambda_2}(s) + \operatorname{Im} A_{\lambda_1\lambda_2}(-s)\right]$$

Now use crossing:

$$A_{\lambda_1\lambda_2\lambda_1\lambda_2}(-s) = A_{\lambda_1\lambda_2\lambda_1\lambda_2}(s)$$

$$\frac{d^2 A_{\lambda_1\lambda_2}}{ds^2}(0) = \frac{4}{\pi}\int_{4m^2}^{+\infty}\frac{ds}{s^3}\operatorname{Im} A_{\lambda_1\lambda_2}(s)$$

Now use the optical theorem:

$$\operatorname{Im} A_{\lambda_1\lambda_2}(s) = s\,\sigma_{\lambda_1\lambda_2\to \text{all}}(s) \equiv s\,\sigma_{\lambda_1\lambda_2}(s)$$

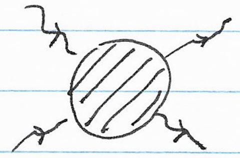
$$\frac{d^2 A_{\lambda_1\lambda_2}}{ds^2}(0) = \frac{4}{\pi}\int_{4m^2}^{+\infty}\frac{ds}{s^2}\sigma_{\lambda_1\lambda_2}(s) > 0$$

Apply this to $A_{xx} = A_{xxxx} = \frac{s^2}{2}(2a_1 + a_2)$

and $A_{xy} = A_{xyxy} = \frac{s^2}{4}a_2$

$\rightarrow$ $$\boxed{\begin{array}{c} 2a_1 + a_2 > 0 \\ a_2 > 0 \end{array}}$$

Process: forward scattering amplitude of light by light scattering with fixed photon polarization:



here λ_i denote the γ polarization
They can describe γ helicities : $\lambda_i = \pm 1$
or linear polarizations: $\lambda_i = (x, y)$

$$A_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(s, t, u) = \epsilon^{*\alpha_4}_{\lambda_4} \epsilon^{*\alpha_3}_{\lambda_3} \epsilon^{\alpha_2}_{\lambda_2} \epsilon^{\alpha_1}_{\lambda_1}$$
$$\mathcal{M}_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}(k_1, k_2, k_3, k_4)$$

$k_i^2 = 0 \qquad k_{i\mu} \epsilon^{\mu}_{\lambda}(k_i) = 0$

$s = (k_1 + k_2)^2 \qquad t = (k_1 - k_3)^2 \qquad u = (k_1 - k_4)^2$

$s + t + u = 0$

$k_1 + k_2 = k_3 + k_4$

Forward scattering : $k_3 = k_1$ $\quad$ $\downarrow$ $k_2 = k_4$

$t = 0 \qquad u = -s$

$\rightarrow A_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(s)$

we have seen that

$$A_{\lambda_1 \lambda_2 \lambda_3 \lambda_4}(s) = [A(s)\, g_{\alpha_1 \alpha_3} g_{\alpha_2 \alpha_4} + B(s)\, g_{\alpha_1 \alpha_4} g_{\alpha_2 \alpha_3} + C(s)\, g_{\alpha_1 \alpha_2} g_{\alpha_3 \alpha_4}] \cdot \epsilon\ \epsilon\ \epsilon\ \epsilon$$