Exercises 22. 11. 2019

We have shown: forward light-by-light scattering

$$A_{\lambda_1\lambda_2\lambda_3\lambda_4}(s) = \epsilon^{*\alpha_4}_{\lambda_4}(k_2)\,\epsilon^{*\alpha_3}_{\lambda_3}(k_1)\,\epsilon^{\alpha_2}_{\lambda_2}(k_2)\,\epsilon^{\alpha_1}_{\lambda_1}(k_1)$$

$$\left[A(s)\,g_{\alpha_1\alpha_3}\,g_{\alpha_2\alpha_4} + B(s)\,g_{\alpha_1\alpha_4}\,g_{\alpha_2\alpha_3} + C(s)\,g_{\alpha_1\alpha_2}\,g_{\alpha_3\alpha_4}\right]$$

By imposing

$$A_{-h_3h_2-h_1h_4}(-s) = A_{h_1h_2h_3h_4}(s)$$

$$A(-s) = A(s)$$
$$C(-s) = B(s)$$

Now go to linear polarization:

$\epsilon_x(k_1) = (0,1,0,0)$ $\qquad$ $\epsilon_x(k_2) = (0,-1,0,0)$

$\epsilon_y(k_1) = (0,0,1,0)$ $\qquad$ $\epsilon_y(k_2) = (0,0,1,0)$

$$\epsilon^*_{\lambda_3}(k_1)\,\epsilon_{\lambda_1}(k_1)\;\epsilon^*_{\lambda_4}(k_2)\,\epsilon_{\lambda_2}(k_2) = \delta_{\lambda_1\lambda_3}\,\delta_{\lambda_2\lambda_4}$$

$$\epsilon^*_{\lambda_4}(k_2)\,\epsilon_{\lambda_1}(k_1)\;\epsilon^*_{\lambda_3}(k_1)\,\epsilon_{\lambda_2}(k_2) = (\cancel{-1})\,\delta_{\lambda_1\lambda_4}\,(\cancel{-1})\,\delta_{\lambda_2\lambda_3}$$

$$\epsilon_{\lambda_2}(k_2)\,\epsilon_{\lambda_1}(k_1)\;\epsilon^*_{\lambda_4}(k_2)\,\epsilon^*_{\lambda_3}(k_1) = (\cancel{-1})\,\delta_{\lambda_1\lambda_2}\,(\cancel{-1})\,\delta_{\lambda_3\lambda_4}$$

$$A_{\lambda_1\lambda_2\lambda_3\lambda_4}(s) = A(s)\,\delta_{\lambda_1\lambda_3}\,\delta_{\lambda_2\lambda_4} + B(s)\,\delta_{\lambda_1\lambda_4}\,\delta_{\lambda_2\lambda_3} + B(-s)\,\delta_{\lambda_1\lambda_2}\,\delta_{\lambda_3\lambda_4}$$

$$A_{xxxx}(s) = A_{yyyy}(s) = A(s) + B(s) + B(-s)$$
$$A_{xyxy}(s) = A_{yxyx}(s) = A(s)$$
$$A_{xyyx}(s) = A_{yxxy}(s) = B(s)$$
$$A_{xxyy}(s) = A_{yyxx}(s) = B(-s)$$

$$\to \begin{cases} A_{xxxx}(-s) = A_{xxxx}(s) \\ A_{xyxy}(-s) = A_{xyxy}(s) \end{cases} \quad + (x \leftrightarrow y)$$

$$A_{xyyx}(-s) = A_{yyxx}(s)$$

Exercise derive the positivity constraint for the IR theory corresponding to $\mathcal{L}_{UV}$:

$$\mathcal{L}_{UV} = \partial_\mu \varphi^+ \partial^\mu \varphi - V(\varphi^+\varphi)$$
$$V(\varphi^+\varphi) = \lambda\left(\varphi^+\varphi - \frac{v^2}{2}\right)^2$$

U(1) global symmetry : $\varphi \to e^{-i\alpha}\varphi$
minimum of V at $|\varphi|^2 = \frac{v^2}{2}$
U(1) spontaneously broken $\to$ Goldstone boson ξ
Parametrize

$$\varphi = \frac{(\sigma + v)}{\sqrt{2}} e^{i\frac{\xi}{v}}$$

$$\varphi^+\varphi - v^2 = \frac{(\sigma+v)^2 - v^2}{2}$$

$$V = \frac{\lambda}{4}(\sigma^2 + 2v\sigma)^2 \quad \to \begin{cases} \xi \text{ massless} \\ m_\sigma^2 = 2\lambda v^2 \end{cases}$$

$$\partial_\mu \varphi^+ \partial_\mu \varphi = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu\sigma \cdot \cancel{\sigma}\, \partial^\mu e^{i\xi/v} + \frac{1}{2}\partial_\mu\sigma \cdot \cancel{\sigma}\, \partial_\mu e^{-i\xi/v}$$
$$+ \frac{(\sigma+v)^2}{2}\frac{1}{v^2}\partial_\mu\xi\partial^\mu\xi$$

Integrate out σ at tree-level $E \ll m_\sigma$

e.o.m. $\mathcal{L} = -\frac{1}{2}\sigma\Box\sigma + \frac{1}{2}\left(1+\frac{\sigma}{v}\right)^2 \partial_\mu\xi\,\partial^\mu\xi$

$$- \frac{\lambda}{4}\sigma^4 - \lambda v \sigma^3 - \underbrace{\lambda v^2}_{\frac{m_\sigma^2}{2}}\sigma^2$$

$$\lambda v = \frac{\lambda v^2}{v} = \frac{m_\sigma^2}{2v} \qquad \lambda = \frac{\lambda v^2}{v^2} = \frac{m_\sigma^2}{2v^2}$$

$$\mathcal{L} = -\frac{1}{2}\sigma\Box\sigma + \frac{1}{2}\left(1+\frac{\sigma}{v}\right)^2 \partial_\mu\xi\,\partial^\mu\xi$$

$$- \frac{1}{2}m_\sigma^2\sigma^2 - \frac{1}{2v}m_\sigma^2\sigma^3 - \frac{1}{8}\frac{m_\sigma^2}{v^2}\sigma^4$$

e.o.m. in the static limit $\sigma\Box\sigma \simeq 0$

$$\frac{1}{v}\left(1+\frac{\sigma}{v}\right)\partial_\mu\xi\,\partial^\mu\xi - m_\sigma^2\sigma - \frac{3m_\sigma^2}{2v}\sigma^2 - \frac{1}{2}\frac{m_\sigma^2\sigma^3}{v^2} = 0$$

solve iteratively in the limit $\frac{m_\sigma}{v} \ll 1$

$$\frac{1}{v}\left(1+\frac{\sigma}{v}\right)\partial_\mu\xi\,\partial^\mu\xi - m_\sigma^2\sigma = 0$$

$$\left(\frac{1}{v^2}\partial_\mu\xi\,\partial^\mu\xi - m_\sigma^2\right)\sigma = -\frac{\partial_\mu\xi\,\partial^\mu\xi}{v}$$

$$\boxed{\sigma = \frac{1}{v}\partial_\mu\xi\,\partial^\mu\xi\,\frac{1}{m_\sigma^2\left(1-\frac{\partial_\mu\xi\,\partial^\mu\xi}{v^2 m_\sigma^2}\right)} + \cdots}$$

Expanding:

$$\sigma = \frac{1}{v\,m_\sigma^2}\partial_\mu\xi\,\partial^\mu\xi\left(1 + \frac{\partial_\mu\xi\,\partial^\mu\xi}{v^2 m_\sigma^2} + \cdots\right)$$

$$\mathcal{L}_{IR} = \frac{1}{2}\left(1 + \frac{\partial_\mu \xi \partial^\mu \xi}{v^2 m_\sigma^2} + O\left(\frac{1}{v^4 m_\sigma^4}\right)\right)^2 \partial_\mu \xi \partial^\mu \xi$$

$$- \frac{1}{2} \cancel{m_\sigma^2} \frac{1}{v^2 m_\sigma^{\cancel{4}\,2}} (\partial_\mu \xi \partial^\mu \xi)^2 + \dots$$

$$\boxed{= \frac{1}{2} \partial_\mu \xi \partial^\mu \xi + \frac{1}{2} \frac{(\partial_\mu \xi \partial^\mu \xi)^2}{v^2 m_\sigma^2} + \dots = \mathcal{L}_{IR}}$$

Bottom-up consideration:
under $U(1)$ $\xi \to \xi$ [redacted] $v\alpha$ shift symmetry
therefore $\mathcal{L}_{IR}$ can only depend on $\partial_\mu \xi$
and we could have written:

$$\mathcal{L}_{IR} = \frac{1}{2} \partial_\mu \xi \partial^\mu \xi + \frac{c_1}{\Lambda^4} (\partial_\mu \xi \partial^\mu \xi)^2 + \frac{c_2}{\Lambda^2} (\Box \xi \cdot \Box \xi) + \dots$$

redundant

power counting: each ξ comes in the combination $\frac{\xi}{v}$: $c_1 \simeq \frac{1}{v^2 \Lambda^2}$

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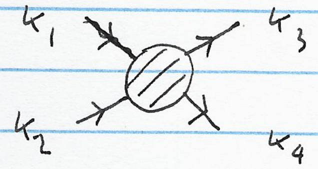
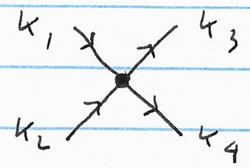
The coefficient of $(\partial_\mu \xi \partial^\mu \xi)^2$ is <u>positive</u>
Show that this is a consequence of analyticity, crossing and optical theorem.

Parity : $\varphi \xrightarrow{P} \varphi^\dagger$ $\xi \to -\xi$

Forward scattering amplitude for $\xi\xi \to \xi\xi$



$= \mathrm{IR} =$



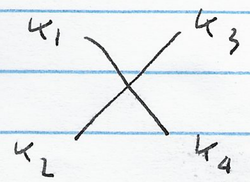
$+$ loops

Feynman rule from $\partial_\mu\xi\,\partial^\mu\xi\;\partial_\nu\xi\,\partial^\nu\xi$

proportional to:

$$A\; k_1k_3\;\; k_2k_4 + B\; k_1k_4\;\; k_2k_3 + C\; k_1k_2\;\; k_3k_4$$

Explicit computation gives $A = B = C = 8$



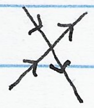
$$= 8i\left(k_1k_3\;\; k_2k_4 + k_1k_4\;\; k_2k_3 + k_1k_2\;\; k_3k_4\right)\frac{c_1}{\Lambda^4}$$

$s = (k_1+k_2)^2 = 2k_1k_2$

$t = (k_1-k_3)^2 = -2k_1k_3$

$u = (k_1-k_4)^2 = -2k_1k_4$

$s+t+u = 2k_1(\underbrace{k_2-k_3-k_4}_{\equiv -k_1}) \equiv 0$



$$= \frac{8i}{\Lambda^4}\cdot\frac{1}{4}\left(t^2+u^2+s^2\right)c_1$$

Forward scattering $t = 0$ $u = -s$

$$iA(s) = \frac{4i}{\Lambda^4}\, s^2 c_1 \qquad \text{in IR region} \quad s \ll m_\sigma^2$$

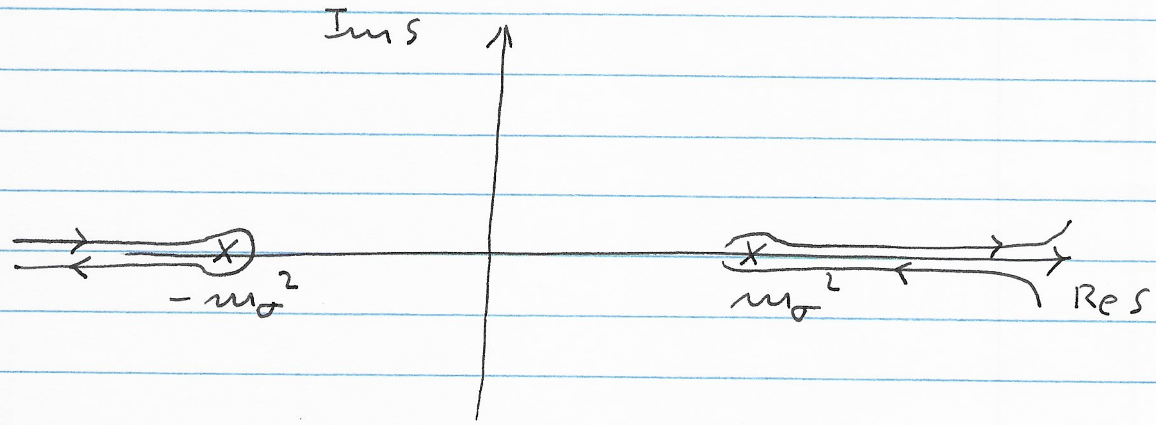
$$A(s) = \frac{4c_1}{\Lambda^4}\, s^2$$

Singularities of $A(s)$ in the complex plane:

crossing symmetry $A(-s) = A(s)$

$A(s)$ has a pole at $s = m_\sigma^2$

+ crossing $\rightarrow$ another pole at $s = -m_\sigma^2$



<u>Dispersion relation</u>

$$\left.\frac{d^2A}{ds^2}\right|_{s=0} = \frac{1}{\pi i} \oint \frac{ds}{s^3} A(s)$$

$=$ contribution at ∞ $\rightarrow 0$ if $\left|\frac{A(s)}{s}\right| \rightarrow 0$

$$= \frac{1}{\pi i}\Big[-\underbrace{\oint \frac{ds}{s^3} A(s)}_{\substack{\text{around} \\ s = m_\sigma^2}} - \underbrace{\oint \frac{ds}{s^3} A(s)}_{\substack{\text{around} \\ s = -m_\sigma^2}} \Big]$$

$s = m_\sigma^2 \quad \rightarrow \quad s \equiv m_\sigma^2 + \epsilon e^{i\vartheta} \qquad ds = i\epsilon e^{i\vartheta} d\vartheta$

$$\underset{\substack{\text{around} \\ s = m_\sigma^2}}{\oint \frac{ds}{s^3} A(s)} = i\epsilon \int_0^{2\pi} \frac{d\vartheta\, e^{i\vartheta}}{(m_\sigma^2)^3} \; \frac{\operatorname{Res} A(m_\sigma^2)}{\underbrace{s - m_\sigma^2}_{\epsilon e^{i\vartheta}} + i\epsilon}$$

$$\left.\frac{d^2A}{ds^2}\right|_{s=0} = \frac{-1}{\cancel{\pi i}} \, 2\cancel{\pi i} \left[\frac{\mathrm{Res}\, A(m_\sigma^2)}{(m_\sigma^2)^3} + \frac{\mathrm{Res}\, A(-m_\sigma^2)}{(m_\sigma^2)^3} \right]$$

But near $s = m_\sigma^2$

$$A(s) = \frac{\mathrm{Res} A m_\sigma^2)}{s - m_\sigma^2 + i\epsilon} = \frac{\mathrm{Res} A m_\sigma^2)}{(s - m_\sigma^2)^2 + \epsilon^2} [(s - m_\sigma^2) - i\epsilon]$$

$$\rightarrow \mathrm{Im}\, A(s) = - \mathrm{Res} A m_\sigma^2) \underbrace{\frac{\epsilon}{(s - m_\sigma^2)^2 + \epsilon^2}}_{\hookrightarrow \pi \delta(s - m_\sigma^2)}$$

$$= - \mathrm{Res} A m_\sigma^2) \, \pi \, \delta(s - m_\sigma^2)$$

$$\frac{\mathrm{Res}\, A(m_\sigma^2)}{(m_\sigma^2)^3} = \int ds \, \frac{\mathrm{Res}\, A(m_\sigma^2)}{s^3} \, \delta(s - m_\sigma^2)$$

$$= -\frac{1}{\pi} \int ds \, \frac{\mathrm{Im}\, A(s)}{s^3}$$

$$\left.\frac{d^2A}{ds^2}\right|_{s=0} = \frac{2}{\pi} \int ds \left[\frac{\mathrm{Im}\, A(s)}{s^3} + \frac{\mathrm{Im}\, A(-s)}{s^3} \right]$$

$$= \frac{4}{\pi} \int ds \, \frac{\mathrm{Im}\, A(s)}{s^3}$$

= optical theorem

$$= \frac{4}{\pi} \int \frac{ds}{s^2} \sum_x \sigma_{ss \to x}(s) > 0$$

computing $\left.\frac{d^2 A}{ds^2}\right|_{s=0}$ in the IR we get

$$\frac{8 c_1}{\Lambda^4} = \frac{4}{\pi} \int \frac{ds}{s^2} \sum_X \sigma_{\phi\phi \to X}(s) > 0$$

$$\boxed{c_1 > 0}$$