Loop corrections II

We have seen that loops are / can be responsible of new operators in $\mathcal{L}_{IR}$, not arising at the tree-level.

Another effect of loops is, in general, that of modifying the scaling behaviour of the operators.

At the tree-level, in a relativistic QFT in d space-time dimensions

$$O_i \to e^{(d-[O_i])\alpha} O_i$$

as

$$x = e^{\alpha} \underbrace{x'}_{\text{fixed}}$$

$\alpha \to +\infty$ large distance (small E)

We will study this effect on a concrete example. The UV theory, defined in $d=4$, contains 2 real scalar fields φ and H and has a Z_2 symmetry acting on φ as

$$\varphi \to -\varphi$$

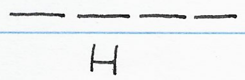
Therefore

$$\mathcal{L}_{UV} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_L^2\varphi^2 + \frac{1}{2}\partial_\mu H\partial^\mu H - \frac{1}{2}M^2H^2$$

$$-\frac{\lambda_0}{4!}\varphi^4 - \frac{\lambda_1}{2}M\varphi^2 H - \frac{\lambda_2}{4}\varphi^2H^2 + \dots$$

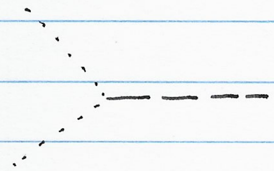
H^3, H^4 terms neglected here

Remark $\varphi^2 H$ has a dimensionful parameter $\sim (\lambda_1 M)$. If $\lambda_1 \approx O(1)$ this affects the power counting of the IR theory since powers of M can now come both from



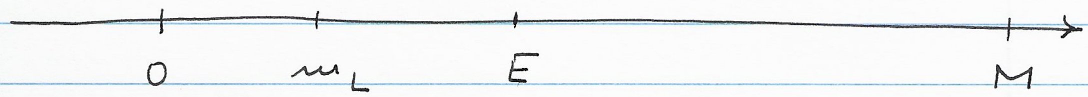
$$\frac{i}{k^2 - M^2}$$

and



$$-i \lambda_1 M$$

we assume $M \gg E \gg m_L$



Task: find $\mathcal{L}_{IR}$ valid in this energy range

Notice: we are temporarily assuming that, after inclusion of the relevant effects, $m_\varphi \approx m_L$

In a bottom-up perspective, we have:

$$\mathcal{L}_{IR} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - C_4 \frac{\varphi^4}{4!} - \frac{C_6}{M^2} \frac{\varphi^6}{6!} + O\left(\frac{1}{M^4}\right)$$

<u>Exercise</u>

Find out additional $d = 6$ operators and show that they are redundant, up to total derivatives

$$O_{6,1} = \Box\varphi \, \Box\varphi$$
$$O_{6,2} = \varphi \, \Box \varphi^3$$
$$O_{6,3} = \varphi^2 \Box \varphi^2$$
$$O_{6,4} = \varphi^2 \partial_\mu \varphi \partial^\mu \varphi$$

$\vdots$

$O_{6,3}$ and $O_{6,4}$ can be expressed in terms of $O_{6,2}$ by partial integration:

$$\varphi^2 \partial_\mu \varphi \partial^\mu \varphi = - \varphi \partial_\mu \varphi^2 \cdot \partial^\mu \varphi - \varphi^3 \Box \varphi$$
$$= -2 \varphi^2 \partial_\mu \varphi \partial^\mu \varphi - \varphi^3 \Box \varphi$$

$$\rightarrow \boxed{3 \varphi^2 \partial_\mu \varphi \partial^\mu \varphi = - \varphi^3 \Box \varphi \approx - \varphi \Box \varphi^3}$$

$$\varphi^2 \Box \varphi^2 = \varphi^2 \partial_\mu \partial^\mu \varphi^2 = - \partial_\mu \varphi^2 \cdot \partial^\mu \varphi^2$$
$$= -4 \varphi^2 \partial_\mu \varphi \partial^\mu \varphi$$
$$= + \frac{4}{3} \varphi \Box \varphi^3$$

$$\boxed{\varphi^2 \Box \varphi^2 = \frac{4}{3} \varphi \Box \varphi^3}$$

We are left with $O_{6,1}$ and $O_{6,2}$

$$\mathcal{L}'_{IR} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m'^2 \varphi^2 - \frac{c_4' \varphi^4}{4!}$$
$$- \frac{c_6'}{M^2} \frac{\varphi^6}{6!} - \frac{\tilde{c}_6}{M^2} \frac{\varphi^3 \Box \varphi}{4!} - \frac{\hat{c}_6}{M^2} \frac{\Box \varphi \Box \varphi}{2}$$

EOM:
$$(\Box + m'^2) \varphi + \frac{c_4'}{6} \varphi^3 = O(M^{-2})$$
$$\Box \varphi = - m'^2 \varphi - \frac{c_4'}{6} \varphi^3 + O(M^{-2})$$

$$\mathcal{L}'_{IR} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m'^2\varphi^2 - c_4'\frac{\varphi^4}{4!} - \frac{c_6'}{M^2}\frac{\varphi^6}{6!}$$
$$+ \frac{\tilde{c}_6}{M^2}\frac{\varphi^3}{4!}\left(m'^2\varphi + \frac{c_4}{6}\varphi^3\right)$$
$$- \frac{\hat{c}_6}{M^2}\frac{1}{2}\left(m'^2\varphi + \frac{c_4}{6}\varphi^3\right)^2 + O(M^{-4})$$

$$= \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m'^2\varphi^2 - \frac{1}{2}\left(\frac{m'^4}{M^2}\hat{c}_6\right)\varphi^2$$
$$- \frac{\varphi^4}{4!}\left(c_4' - \frac{m'^2}{M^2}\tilde{c}_6 + 4\frac{m'^2}{M^2}\hat{c}_6 c_4'\right)$$
$$- \frac{\varphi^6}{6!}\frac{1}{M^2}\left(c_6' - 5\tilde{c}_6 c_4' + 10\hat{c}_6 c_4'^2\right)$$

Relation between $\mathcal{L}_{IR}$ and $\mathcal{L}'_{IR}$

$$\begin{cases} c_4 = c_4' - \dfrac{m'^2}{M^2}\tilde{c}_6 + 4\dfrac{m'^2}{M^2}\hat{c}_6 c_4' \\ c_6 = c_6' - 5\tilde{c}_6 c_4' + 10\hat{c}_6 c_4'^2 \\ m^2 = m'^2 + \dfrac{m'^4}{M^2}\hat{c}_6 \end{cases}$$

If $\hat{c}_6 = 0$

$$\begin{cases} m^2 = m'^2 \\ c_4 = c_4' - \dfrac{m'^2}{M^2}\tilde{c}_6 \\ c_6 = c_6' - 5\tilde{c}_6 c_4' \end{cases}$$

~~[illegible]~~

We can use this to trade the operator φ^6 for $\varphi^3\Box\varphi \approx \varphi\Box\varphi^3$

unboxed basis   boxed basis

$$\mathcal{L}_{IR}(m^2, c_4, c_6) \longleftrightarrow \mathcal{L}'_{IR}(m'^2, c_4', \tilde{c}_6)$$

$\to$ set $c_6' = 0$

$$\begin{cases} m'^2 = m^2 \\ c_4 = c_4' - \dfrac{m^2}{M^2}\tilde{c}_6 \\ c_6 = -5\,\tilde{c}_6\, c_4' \end{cases}$$

We can invert this, to 1st order in $\frac{m^2}{M^2}$

0th order $c_4' = c_4$

$$c_6 = -5\,\tilde{c}_6\, c_4'$$

$\to$ $\tilde{c}_6 = -\dfrac{c_6}{5c_4}$ $c_4' = c_4$

$c_4' = c_4 + \delta_4$ $\tilde{c}_6 = -\dfrac{c_6}{5c_4} + \delta_6$

$$\cancel{c_4} = \cancel{c_4} + \delta_4 + \frac{m^2}{M^2}\frac{c_6}{5c_4} + \ldots$$

$$\to \delta_4 = -\frac{m^2}{M^2}\cdot\frac{c_6}{5c_4}$$

$$\cancel{c_6} = -5\left(-\frac{\cancel{c_6}}{\cancel{5}c_4} + \delta_6\right)(c_4 + \delta_4)$$

$-5\,\delta_6 c_4 \approx 0$ $\delta_6 = 0$

$$\boxed{c_4' = c_4 - \frac{m^2}{M^2}\frac{c_6}{5c_4} \qquad \tilde{c}_6 = -\frac{c_6}{5c_4}}$$

Tree-level matching

E.O.M. of the heavy mode:

$$(\Box + M^2)H + \frac{\lambda_1}{2} M\varphi^2 + \frac{\lambda_2}{2}\varphi^2 H = 0$$

$$\left(\Box + M^2 + \frac{\lambda_2}{2}\varphi^2\right) H = -\frac{\lambda_1}{2} M\varphi^2$$

solution (also keeping the box):

$$H_c = -\frac{\lambda_1}{2} M \left(\Box + M^2 + \frac{\lambda_2}{2}\varphi^2\right)^{-1}\varphi^2$$

Back into the $\mathcal{L}_{UV}$:

$$\mathcal{L}_{IR} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_L^2\varphi^2 - \frac{\lambda_0}{4!}\varphi^4$$

$$-\frac{1}{2} H_c(\varphi)\left(\Box + M^2 + \frac{\lambda_2}{2}\varphi^2\right)H_c(\varphi)$$

$$-\frac{\lambda_1}{2} M\varphi^2 H_c(\varphi)$$

$$= \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_L^2\varphi^2 - \frac{\lambda_0}{4!}\varphi^4$$

$$-\frac{1}{2}\frac{\lambda_1^2}{4} M^2\varphi^2\left(\Box + M^2 + \frac{\lambda_2}{2}\varphi^2\right)^{-1}\varphi^2$$

$$+\frac{\lambda_1}{2}\frac{\lambda_1}{2} M^2\varphi^2\left(\Box + M^2 + \frac{\lambda_2}{2}\varphi^2\right)^{-1}\varphi^2$$

$$= \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_L^2\varphi^2 - \frac{\lambda_0}{4!}\varphi^4$$

$$+\frac{\lambda_1^2}{8} M^2\varphi^2\left(\Box + M^2 + \frac{\lambda_2}{2}\varphi^2\right)^{-1}\varphi^2$$

By expanding

$$\left(\Box + M^2 + \frac{\lambda_2}{2}\varphi^2\right)^{-1} = \frac{1}{M^2}\left(1 + \frac{\Box}{M^2} + \frac{\lambda_2}{2}\frac{\varphi^2}{M^2}\right)^{-1}$$

$$\simeq \frac{1}{M^2}\left(1 - \frac{\Box}{M^2} - \frac{\lambda_2}{2}\frac{\varphi^2}{M^2} + \dots\right)$$

we get:

$$\mathcal{L}_{IR} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_L^2\varphi^2 - \frac{\lambda_0}{4!}\varphi^4$$

$$+ \frac{\lambda_1^2}{8}\varphi^2\left(1 - \frac{\Box}{M^2} - \frac{\lambda_2}{2}\frac{\varphi^2}{M^2}\right)\varphi^2 + \dots$$

$$\mathcal{L}_{IR} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_L^2\varphi^2 - \frac{\lambda_0}{4!}\varphi^4$$

$$+ \frac{\lambda_1^2}{8}\varphi^4 - \frac{\lambda_1^2}{8}\cdot\frac{1}{M^2}\varphi^2\Box\varphi^2 - \frac{\lambda_1^2\lambda_2}{16}\frac{1}{M^2}\varphi^6 + \dots$$

$$\mathcal{L}_{IR} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_L^2\varphi^2$$

$$- \frac{\varphi^4}{4!}\left(\lambda_0 - 3\lambda_1^2\right)$$

$$- \frac{\varphi^6}{6!\,M^2}\cdot \lambda_1^2\lambda_2 \times \underbrace{\frac{6!}{16}}_{45}$$

$$- \frac{\lambda_1^2}{8M^2}\underbrace{\varphi^2\Box\varphi^2}_{\frac{4}{3}\varphi^3\Box\varphi}$$

As we have already seen, the last operator is redundant and can be eliminated using e.o.m.

$$(\Box + m_L^2)\varphi + \frac{(\lambda_0 - 3\lambda_1^2)}{4!}\, 4\varphi^3 = O(M^{-2})$$

$$\varphi^3 \Box \varphi = -\varphi^3\left(m_L^2\varphi + \frac{(\lambda_0 - 3\lambda_1^2)\varphi^3}{6} + O(M^{-2})\right)$$

$$= -m_L^2\varphi^4 - \frac{(\lambda_0 - 3\lambda_1^2)}{6}\varphi^6 + O(M^{-2})$$

The last term becomes

$$+ \frac{\cancel{4}}{3}\,\frac{\lambda_1^2}{{}_2\cancel{8}M^2}\left(m_L^2\varphi^4 + \frac{(\lambda_0 - 3\lambda_1^2)}{6}\varphi^6\right) + \dots$$

$$= \frac{\lambda_1^2}{6}\,\frac{m_L^2}{M^2}\varphi^4 + \frac{(\lambda_0 - 3\lambda_1^2)\lambda_1^2}{36\,M^2}\varphi^6 + \dots$$

We end up with:

$$\boxed{\begin{aligned}\mathcal{L}_{IR} &= \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}m_L^2\varphi^2 \\ &\quad - \frac{\varphi^4}{4!}\left(\lambda_0 - 3\lambda_1^2 - 4\lambda_1^2\frac{m_L^2}{M^2}\right) \\ &\quad - \frac{\varphi^6}{6!\,M^2}\left(45\lambda_1^2\lambda_2 - \underbrace{20\lambda_0\lambda_1^2 + 60\lambda_1^4}_{\text{I would have missed these by neglecting the } \Box}\right) + \dots\end{aligned}}$$