Loops change the scaling properties of operators

→ General remark, not specific of EFT

Example:

$$\mathcal{L}(\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

now seen either as a UV or an IR EFT.

Naive scaling in $d=4$: $\quad x = e^{\alpha}x'$, fixed

$$\varphi^4 \longrightarrow e^{d-[\varphi^4]}\varphi^4 \equiv \varphi^4 \qquad \underline{\text{marginal}}$$

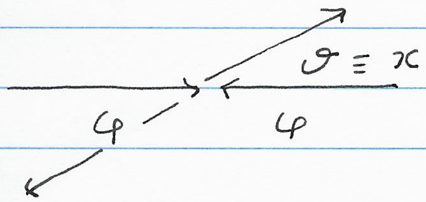
Another way of saying this. Consider the amplitude for $\varphi\varphi \to \varphi\varphi$

~~At the~~ TL

~~$A(\varphi\varphi\to\varphi\varphi) = -i\lambda$ dimensionless~~

In principle $\quad A = A(E, x, \lambda, m)$

↳ dimensionless variables arising from kinematics such as angles etc...



We know that $[A(\varphi\varphi\to\varphi\varphi)] = 0$

So we expect:

$$A(E, x, \lambda, m) = E^0 A(1, x, \lambda, \frac{m}{E})$$

more generally if $[A] = D$

$$A(E, x, \lambda, m) = E^D A\left(1, x, \lambda, \frac{m}{E}\right)$$

If $E \gg m$

we naively expect:

$$A(E, x, \lambda, m) \xrightarrow{E \to \infty} E^D A(1, x, \lambda, 0)$$

For $\varphi\varphi \to \varphi\varphi$

$$A(E, x, \lambda, m) \xrightarrow{E \to \infty} A(1, x, \lambda, 0)$$

This is indeed true at the TL:

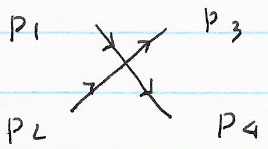
$$\boxed{A(\varphi\varphi \to \varphi\varphi) = -i\lambda \qquad (1)}$$

but it is no more true at 1-loop order, as we now see.

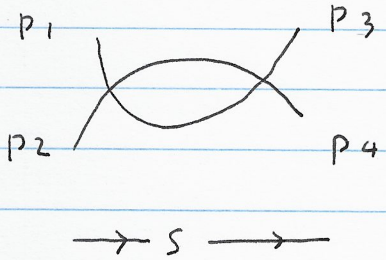
Notice that (1) is another way of saying that φ^4 is marginal: no E dependence.

What happens at 1-loop?

Now we have several diagrams

TL  $= -i\lambda$

1 - loop :



+ t-channel + u-channel

(1) Compute the s-channel in <u>dimensional reg.</u>
We move from 4 to d dimension and evaluate the integrals in $d \neq 4$. d is seen as a <u>complex</u> variable. Defining

$$\varepsilon \equiv 4 - d$$

Divergences are recovered in the limit $\epsilon \to 0$
As long as $\epsilon \neq 0$ integrals are finite

Action $= \int d^d x \, \mathcal{L}$ $\rightarrow$ now $[\mathcal{L}] = d$

$\rightarrow [\varphi] = \frac{d-2}{2}$

$[m^2 \varphi^2] = d = [m^2] + d - 2 \rightarrow [m^2] = 2$

$[\lambda \varphi^4] = d = [\lambda] + 2d - 4 \rightarrow [\lambda] = 4 - d = \epsilon$

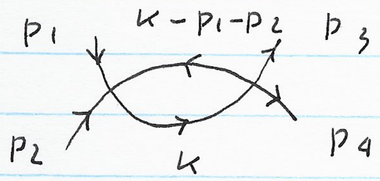
Better make λ adimensional by writing

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \mu^\epsilon \frac{\lambda}{4!} \varphi^4$$

now adimens.

$[\mu] = +1$ μ is called ~~a~~ t'Hoot parameter

$$\times = -i\lambda\mu^{\epsilon} \qquad \frac{i}{k^2-m^2+i\epsilon}$$



$$= (-i\lambda)^2\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\frac{i^2}{(k^2-m^2)((k-\underbrace{p_1-p_2}_{\equiv p})^2-m^2)} \times\left(\frac{1}{2}\right)$$

symmetry factor

② Feynman trick:

$$\frac{1}{AB} = \int_0^1 dx\,\frac{1}{[Ax+B(1-x)]^2}$$

$$\frac{1}{((k-p)^2-m^2)(k^2-m^2)} =$$

$$= \int_0^1 dx\,\frac{1}{[(k-p)^2x - \cancel{m^2x} + k^2 - m^2 - \cancel{k^2x} + \cancel{m^2x}]^2}$$

$$\cancel{k^2x} - 2kpx + p^2x$$

$$= \int_0^1 dx\,\frac{1}{[(k-px)^2 \underbrace{-p^2x^2+p^2x-m^2}_{-(-p^2x(1-x)+m^2)}]^2}$$

define $\Omega \equiv m^2 - p^2x(1-x)$

$$= \int_0^1 dx\,\frac{1}{[(k-px)^2-\Omega]^2}$$

$$= +\frac{\lambda^2}{2}\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\,\frac{1}{[(k-px)^2-\Omega]^2}$$

③ shift $k \to k + px$

$$+ \frac{\lambda^2}{2} \mu^{2\epsilon} \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \cdot \frac{1}{[k^2 - \Omega]^2}$$

③ Wick rotation to go into Euclidean space

$k_0 \to i k_0 \qquad k^2 \to -k^2 \qquad d^d k \to i\, d^d k$

$$= + i \frac{\lambda^2}{2} \mu^{2\epsilon} \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \Omega)^2}$$

④ $$I(n) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + \Omega)^n} =$$

$$= \frac{1}{(4\pi)^{d/2}} \frac{\Omega^{d/2 - n}}{\cancel{\Gamma(d/2)}} \frac{\Gamma(n - d/2)\cancel{\Gamma(d/2)}}{\Gamma(n)}$$

$I(n)$ expressed in terms of Euler function:

$$\Gamma(z) = \int_0^{+\infty} dt \, t^{z-1} e^{-t} \qquad z \in \mathbb{C}$$

except $z = 0, -1, -2, \ldots$

$\Gamma(z+1) = z\,\Gamma(z) \qquad \Gamma(n) = (n-1)! \quad n \in \mathbb{N}$

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon) \qquad |\epsilon| \ll 1$$

$$\gamma \equiv \left(\sum_1^n {}_k \frac{1}{k} - \log n \right) \Bigg|_{n \to \infty} \simeq 0.5772\ldots$$

Here we need $I(2)$

$$I(2) = \frac{1}{(4\pi)^{d/2}} \Omega^{\frac{d-4}{2}} \frac{\Gamma(\overbrace{\tfrac{4-d}{2}}^{\epsilon/2})}{\underbrace{\Gamma(2)}_{1}}$$

$$= \frac{1}{(4\pi)^2} \cdot \frac{1}{(4\pi)^{-\epsilon/2}} \Omega^{-\epsilon/2} \Gamma(\epsilon/2)$$

Final result TL + 1-loop:

$$-i\lambda\mu^{\epsilon} + i\frac{\lambda^2}{2}\mu^{2\epsilon}\int_0^1 dx \frac{1}{(4\pi)^2}\frac{1}{(4\pi)^{-\epsilon/2}}\Omega^{-\epsilon/2}\Gamma(\epsilon/2)$$

$$+ O(\lambda^3)$$

check dimensions: $[\mu^{2\epsilon}\Omega^{-\epsilon/2}] = 2\epsilon - \epsilon = \epsilon$
o.k.

Consider

$$\mu^{2\epsilon/2}\frac{1}{(4\pi)^{-\epsilon/2}}\Omega^{-\epsilon/2}\Gamma(\epsilon/2)$$

$$= \left(\frac{\Omega}{4\pi\mu^2}\right)^{-\epsilon/2} \underbrace{\Gamma(\epsilon/2)}_{(\frac{2}{\epsilon}-\gamma+\dots)} \approx$$

$$= \left(1 - \frac{\epsilon}{2}\log\frac{\Omega}{4\pi\mu^2}\right)\left(\frac{2}{\epsilon}-\gamma+\dots\right)$$

$$= \frac{2}{\epsilon} - \gamma - \log\frac{\Omega}{4\pi\mu^2} + \dots = \frac{2}{\epsilon} - \gamma + \log 4\pi - \log\frac{\Omega}{\mu^2} + \dots$$

$$x^{-\epsilon/2} = e^{-\frac{\epsilon}{2}\log x} \approx 1 - \frac{\epsilon}{2}\log x$$

TL + 1-loop =

$$= -i\lambda\mu^{\epsilon} + i\frac{\lambda^2}{2\cdot 16\pi^2}\mu^{\epsilon}\int_0^1 dx\left[\underbrace{\frac{2}{\epsilon} - \gamma + \log 4\pi}_{\equiv \underline{\text{Div}}} - \log\frac{\Omega}{\mu^2} + \cdots\right]$$

$$+ \quad t + u$$

renormalized amplitude in $\overline{MS}$ scheme:
remove the (Div) part. At the end let $\epsilon \to 0$

$(T + 1\text{-loop})_{ren}$ =

$$-i\lambda \text{ [redacted] } \frac{i\lambda^2}{2\cdot 16\pi^2}\int_0^1 dx \log\frac{m^2 - p^2 x(1-x)}{\mu^2} + \quad t + u$$

$$= -i\left(\lambda + \frac{\lambda^2}{2\cdot 16\pi^2}\int_0^1 dx\left[\log\frac{m^2 - s\,x(1-x)}{\mu^2}\right.\right.$$

$$+ \log\frac{m^2 - t\,x(1-x)}{\mu^2}$$

$$\left.\left. + \log\frac{m^2 - u\,x(1-x)}{\mu^2}\right]\right)$$

since in CM frame

$s = (p_1+p_2)^2 = 4E^2 \equiv E_{CM}^2$

$t = (p_1-p_3)^2 = -2p^2(1-c_\theta) \equiv -2(E^2-m^2)(1-c_\theta)$

$\underline{u = (p_1-p_4)^2 = -2p^2(1+c_\theta) \equiv -2(E^2-m^2)(1+c_\theta)}$

$s+t+u \qquad\qquad \equiv 4m^2$

$p^2 + m^2 = E^2$

$p_1 = (E, 0, 0, p) \qquad p_3 = (E, p s_\theta, 0, p c_\theta)$

$p_2 = (E, 0, 0, -p) \qquad p_4 = (E, -p s_\theta, 0, -p c_\theta)$

$p_1 - p_3 = (0, -p s_\theta, 0, p(1-c_\theta))$

$(p_1-p_3)^2 = -p^2 s_\theta^2 - p^2(1 - 2c_\theta + c_\theta^2) = -2p^2 + 2p^2 c_\theta = -2p^2(1-c_\theta)$

When we take $E \gg m$ we get

$$-i\left(\lambda + \frac{\lambda^2}{32\pi^2}\left(\log\frac{-s}{\mu^2} + \log\frac{-t}{\mu^2} + \log\frac{-u}{\mu^2}\right)\right)$$

$$\approx -i\left(\lambda + \frac{3\lambda^2}{32\pi^2}\log\frac{E^2}{\mu^2} + \dots\right)$$

$$= -i\lambda\left(1 + \frac{6\lambda}{32\pi^2}\log\frac{E}{\mu} + \dots\right) \sim A(\varphi\varphi \to \varphi\varphi)$$

(1) It is no constant any more.
It grows with the energy $\leftrightarrow$ $\lambda\varphi^4$ becomes slightly irrelevant

Dictionary : $\boxed{\log\frac{E}{\mu} \equiv -\alpha}$

$$1 + \frac{6\lambda}{32\pi^2}(-\alpha) \approx e^{-\frac{6\lambda}{32\pi^2}\alpha} \xrightarrow{\alpha \to +\infty} 0$$

(2) We can define

$$\lambda_{eff}(E) = \lambda\left(1 + \frac{6\lambda}{32\pi^2}\log\frac{E}{\mu}\right)$$

The physics cannot depend on μ
$\to$ λ (the renormalized coupling) has an implicit dependence on μ that cancels the explicit $\log\frac{E}{\mu}$

$$\lambda_{eff}(E) = \lambda(\mu)\left(1 + \frac{6\lambda(\mu)}{32\pi^2}\log\frac{E}{\mu}\right)$$

$$\rightarrow \quad \mu\frac{\partial}{\partial\mu}\lambda_{eff}(E) = 0$$

$$\Updownarrow$$

$$\mu\frac{\partial}{\partial\mu}\lambda(\mu) - \frac{6\lambda^2(\mu)}{32\pi^2} + O\left[\lambda(\mu)\cdot\mu\frac{\partial}{\partial\mu}\lambda\right] \equiv 0$$

$$\beta_\lambda \equiv \mu\frac{\partial}{\partial\mu}\lambda(\mu) = \frac{3\lambda^2}{16\pi^2} + O(\lambda^3)$$

If now we solve:

$$\mu\frac{\partial}{\partial\mu}\lambda = \frac{3\lambda^2}{16\pi^2}$$

consider $\mu\frac{\partial}{\partial\mu}\lambda^{-1} = -\frac{1}{\lambda^2}\mu\frac{\partial}{\partial\mu}\lambda = -\frac{3}{16\pi^2}$

$$\lambda^{-1}(\mu) = \lambda^{-1}(\mu_0) - \frac{3}{16\pi^2}\log\frac{\mu}{\mu_0}$$

$$\lambda(\mu) = \frac{\lambda(\mu_0)}{1 - \frac{3\lambda(\mu_0)}{16\pi^2}\log\frac{\mu}{\mu_0}}$$

$$\approx \lambda(\mu_0) + \frac{3\lambda^2(\mu_0)}{16\pi^2}\log\frac{\mu}{\mu_0}$$

$\rightarrow$ E dependence in $\lambda_{eff}(E)$ is like μ dependence in $\lambda(\mu)$

To study the E-dependence of amplitudes, we can study the μ-dependence of renormalized couplings.

$\divideontimes$

here the dictionary is $\log \frac{\mu}{\mu_0} = -\alpha$

or $\mu = e^{-\alpha} \mu_0$

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