SUMMARY

$$\mathcal{L}_{UV} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m_L^2\varphi^2 + \frac{1}{2}\partial_\mu H\partial^\mu H - \frac{1}{2}M^2H^2$$

$$- \frac{\lambda_0}{4!}\varphi^4 - \frac{\lambda_1}{2}M\varphi^2 H - \frac{\lambda_2}{4}\varphi^2H^2 + \dots$$

2 scalars: φ and H real

Z_2 symmetry: $\varphi \to -\varphi$

In the unboxed basis we have:

$$\mathcal{L}_{IR} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2$$

$$- \frac{\varphi^4}{4!}C_4 - \frac{\varphi^6}{6!}\frac{C_6}{M^2} + \dots$$

$$\begin{cases} m^2 = m_L^2 \\ C_4 = \lambda_0 - 3\lambda_1^2 - 4\lambda_1^2\frac{m_L^2}{M^2} \\ C_6 = 45\lambda_1^2\lambda_2 - 20\lambda_0\lambda_1^2 + 60\lambda_1^4 \end{cases}$$

How this is modified at 1-loop.

We start now

$$m^2 = m_L^2 + O(\hbar)$$

The TL result can also be reproduced by working with the 2-point functions

In the IR:

$$\cdots\cdots (IR) \cdots\cdots = -i(-p^2 + m^2) = i(p^2 - m^2)$$

$$\mathcal{L}_{IR} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \dots = -\frac{1}{2}\varphi(\Box + m^2)\varphi + \dots$$

Similarly in the UV we have

 $= i\,(p^2 - m_L^2)$

$\rightarrow \quad m^2 = m_L^2$

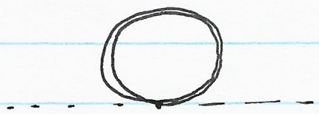
We go on by computing 2-point functions at 1-loop

IR $=$ 

c_4

UV $=$  $+$ 

$\lambda_0 \qquad\qquad \lambda_1^2$

$+$  $+$ 

$\lambda_2 \qquad\qquad \lambda_1^2$

Feynman rules

IR $= \dfrac{i}{k^2 - m^2}$

 $= -i\,c_4\,\mu^{\epsilon}$

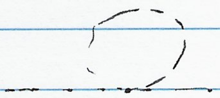
 $-i\lambda_0\mu^\epsilon$

UV

 $= \dfrac{i}{k^2-m_L^2}$  $= \dfrac{i}{k^2-M^2}$

 $= -i\lambda_1 M \mu^{\epsilon/2}$  $= -i\lambda_2\mu^\epsilon$

IR side. Symmetry factor $\equiv \dfrac{1}{4!}\,4\times 3 = \dfrac{1}{2}$

 $= -\dfrac{i c_4}{2}\mu^\epsilon \displaystyle\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2}$

$$= +\frac{c_4}{2}\mu^\epsilon(-i)\underbrace{\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}}_{I(1) = \frac{1}{(4\pi)^{d/2}}(m^2)^{\frac{d}{2}-1}\frac{\Gamma(1-d/2)}{\underbrace{\cancel{\Gamma(1)}}_{1}}}$$

$$= -i\frac{c_4}{2}\mu^\epsilon\frac{1}{(4\pi)^{d/2}}(m^2)^{\frac{d}{2}-1}\left(\frac{2}{2-d}\right)\Gamma\left(\frac{4-d}{2}\right)$$

$$= +i\frac{c_4}{2}\mu^\epsilon\frac{1}{(4\pi)^2}m^2\left(1+\frac{\epsilon}{2}+\dots\right)\frac{1}{(4\pi)^{-\epsilon/2}}(m^2)^{-\epsilon/2}\,\Gamma\left(\frac{4-d}{2}\right)$$

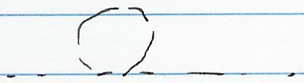
$$= i\frac{c_4}{2}\frac{m^2}{16\pi^2}\left(\frac{2}{\epsilon}-\gamma+\dots\right)\left(1+\frac{\epsilon}{2}+\dots\right)\underbrace{\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon/2}}_{1+\frac{\epsilon}{2}\log\frac{4\pi\mu^2}{m^2}+\dots}$$

$$= i\frac{c_4}{2}\frac{m^2}{16\pi^2}\left(\frac{2}{\epsilon}-\gamma+\log 4\pi+\log\frac{\mu^2}{m^2}+1\right)$$

$[\lambda_1 M] + \frac{3}{2}(d-2) = d$

$[\lambda_1] + 1 - 3 + d\left(\frac{3}{2}-1\right) = 0 \qquad [\lambda_1] = 2 - \frac{d}{2} = \frac{4-d}{2} = \frac{\epsilon}{2}$

SUMMARY IR :



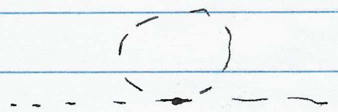
$$= (-i c_4 \mu^\epsilon)\, \frac{1}{2} \int \frac{d^dk}{(2\pi)^d} \cdot \frac{i}{(k^2 - m^2)}$$

$$= -\frac{i c_4}{2} \mu^\epsilon I(1)$$

$$= \left(-i \frac{c_4}{2}\right)\left(-\frac{m^2}{16\pi^2}\right)\left[\mathrm{Div} + \log\frac{\mu^2}{m^2} + 1\right]$$

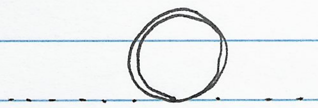
$$\mathrm{Div} = \frac{2}{\epsilon} - \gamma + \log 4\pi$$

UV =



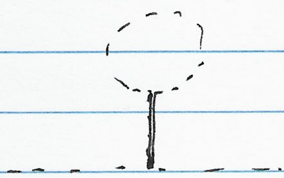
$$= \left(-i \frac{\lambda_0}{2}\right) \mu^\epsilon I(1, m_L)$$

λ_0



$$= \left(-i \frac{\lambda_2}{2}\right) \mu^\epsilon I(1, M)$$

λ_2

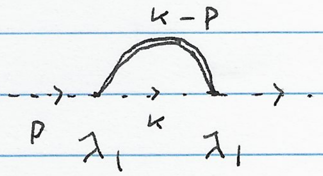


$$= \left(i \frac{\lambda_1^2}{2}\right) \mu^\epsilon I(1, m_L)$$

comes from

$$(-i\lambda_1 M \mu^{\epsilon/2})^2 \frac{i}{(-M^2)} \frac{1}{2} \int \frac{d^dk}{(2\pi)^d} \frac{i}{(k^2 - m_L^2)} = -\frac{\lambda_1^2}{2} \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2 - m_L^2}$$

Finally :



$$= (-i\lambda_1 M)^2 \mu^\epsilon \, i^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 - m_L^2)((k-p)^2 - M^2)}$$

$$= \lambda_1^2 \mu^\epsilon M^2 \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{1}{[(k-p)^2 x - M^2 x + k^2 - m_L^2 - \cancel{k^2 x} + m_L^2 x]^2}$$

$$\cancel{k^2 x} - 2kpx + p^2 x$$

$$= \lambda_1^2 \mu^\epsilon M^2 \int \frac{d^dk}{(2\pi)^d} \int_0^1 dx \frac{1}{[(k-px)^2 + p^2 x(1-x) - m_L^2(1-x) - M^2 x]^2}$$

$$= \lambda_1^2 \mu^\epsilon M^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{[k^2 - \Omega]^2}$$

$$\Omega \equiv m_L^2(1-x) + M^2 x - p^2 x(1-x)$$

$$= i \lambda_1^2 M^2 \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + \Omega)^2}$$

$$= i \lambda_1^2 M^2 \int_0^1 dx \; I(2)$$

$$\hookrightarrow \frac{1}{(4\pi)^2} \left(\frac{4\pi\mu^2}{\Omega}\right)^{\epsilon/2} \Gamma(\epsilon/2)$$

$$= \frac{i \lambda_1^2 M^2}{16\pi^2} \int_0^1 dx \left[1 + \epsilon/2\left(\log 4\pi + \log \frac{\mu^2}{\Omega}\right)\right]\left(\frac{2}{\epsilon} - \gamma + \dots\right)$$

$$= \frac{i \lambda_1^2 M^2}{16\pi^2} \int_0^1 dx \left[\overbrace{\frac{2}{\epsilon} - \gamma + \log 4\pi}^{\text{Div}} + \log \frac{\mu^2}{\Omega}\right]$$

can be evaluated, for further convenience, at $p^2 = m_L^2$ and expanded in powers of $\frac{m_L^2}{M^2}$ leading to:

$$\frac{i}{16\pi^2} \lambda_1^2 M^2 \left[\frac{2}{\epsilon} - \gamma + \log 4\pi + \log \frac{\mu^2}{M^2} + 1 \right.$$

$$+ \cancel{2} \frac{m^2}{M^2} \left(1 + 2 \frac{m^2}{M^2} \right) \log \frac{m^2}{M^2}$$

$$\left. + \frac{1}{2} \frac{m^2}{M^2} + \frac{5}{3} \frac{m^4}{M^4} \right]$$

 $= \dfrac{i\lambda_1^2 M^2}{16\pi^2}\displaystyle\int_0^1 dx \left[\text{Div} + \log \frac{\mu^2}{m_L^2(1-x) + M^2 x - p^2 x(1-x)} \right]$

Now define $\cdots(\text{IR})\cdots$ = tree + loop

and similarly for $\cdots(\text{UV})\cdots$

we have:

$$\cdots(\text{IR})\cdots = i(p^2 - m^2) + i\Pi_{IR}(p^2)$$

$$\cdots(\text{UV})\cdots = i(p^2 - m_L^2) + i\Pi_{UV}(p^2)$$

Physical masses are defined through the equation

$$(\hat{m}^2 - m^2) + \Pi(\hat{m}^2) = 0$$

or $$\hat{m}^2 = m^2 - \Pi(\hat{m}^2)$$
$$\approx m^2 - \Pi(m^2) + O(\hbar^2)$$

We require the two theories to reproduce the same physical mass

$$m^2 - \Pi_{IR}(m^2) = m_L^2 - \Pi_{UV}(m_L^2) + \ldots$$

$\Pi_{IR,UV}(p^2)$ are the renormalized 2-point function. In the $\overline{MS}$ scheme this amounts to remove the Div contribution.

Once we replace C_4 by $\left(\lambda_0 - 3\lambda_1^2 - 4\lambda_1^2 \frac{m_L^2}{M^2}\right)$ in $\Pi_{IR}(m^2)$ we get

$$m^2 = m_L^2 + \Pi_{IR}(m^2) - \Pi_{UV}(m_L^2)$$

↑ here we can replace m with m_L^2

$$\boxed{\begin{aligned} m^2 = m_L^2 &- \frac{1}{32\pi^2}\left[\lambda_2 M^2 + 2\lambda_1^2\left(M^2 + m_L^2 + 2\frac{m_L^4}{M^2}\right)\right] \times \log\frac{\mu^2}{M^2} \\ &- \frac{1}{32\pi^2}\left[\lambda_2 M^2 + \lambda_1^2\left(2M^2 + 3m_L^2 + \frac{22}{3}\frac{m_L^4}{M^2}\right)\right] \end{aligned}}$$

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Remarks

1. Remember that m^2 and m_L^2 depend on μ^2

2. $\log\frac{\mu^2}{m_L^2}$ cancels in the relation.

   This allows to choose $\mu^2 \equiv M^2$ to avoid a breaking of perturbation theory due to a large $\log\frac{\mu^2}{M^2}$, we get:

$$\boxed{\begin{aligned} m^2(M) = m_L^2(M) &- \\ &- \frac{1}{32\pi^2}\left[\lambda_2 M^2 + \lambda_1^2\left(2M^2 + 3m_L^2 + \frac{22}{3}\frac{m_L^4}{M^2}\right)\right] \end{aligned}}$$

This relation fixes the value of $m^2(M)$.
If we want $m^2(\mu)$ for $\mu \ll M$ we exploit the RGE in the IR theory

The physical mass $\tilde{m}^2$ in the IR is:

$$\boxed{\tilde{m}^2 = m^2 - C_4 \frac{m^2}{32\pi^2}\left(\log\frac{\mu^2}{m^2} + 1\right)}$$

The independence of $\tilde{m}$ on μ gives:

$$\mu \frac{d}{\partial \mu}\tilde{m}^2 = 0 \quad \to \quad \frac{d m^2}{d \log \mu} = C_4 \frac{m^2}{16\pi^2} + O(\hbar^2)$$

whose solution is

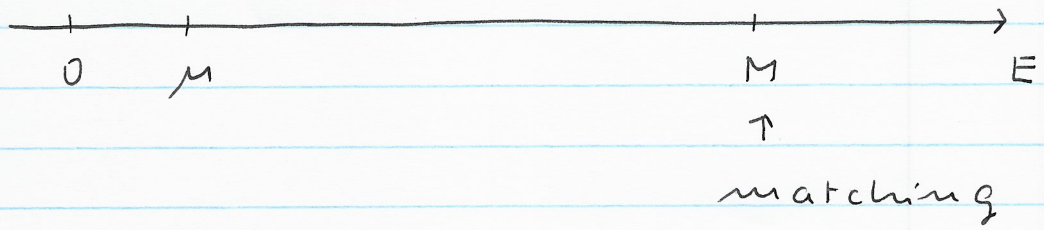
$$m^2(\mu) = m_0^2 \left(\frac{\mu}{\mu_0}\right)^{\frac{C_4}{16\pi^2}}$$

choosing $\mu_0 \equiv M$ we get

$$\boxed{m^2(\mu) = m^2(M)\left(\frac{\mu}{M}\right)^{\frac{C_4}{16\pi^2}}}$$

We have matched the IR and the UV theories at $\mu = M$ and then run down the IR

$$m^2(\mu) \longleftarrow m^2(M) = m_L^2(M) + \dots$$



Recall that the μ dependence is equivalent to the E dependence through the dictionary

$$x^{*} = e^{\alpha}\, \underset{}{\overset{\text{fixed}}{x'}}$$

$$E = e^{-\alpha}\, \underset{\text{fixed}}{E'}$$

$$-\alpha = \log \frac{\mu}{\mu_0}$$

$$m^2(\mu)\,\varphi^2 \longrightarrow m_0^2 \left(\frac{\mu}{\mu_0}\right)^{\frac{c_4}{16\pi^2}} e^{2\alpha}\,\varphi^2$$

$$\equiv e^{2\alpha}\, e^{-\frac{c_4}{16\pi^2}\alpha}\,\varphi^2$$

and the mass operator scales as $\left(2 - \frac{c_4}{16\pi^2}\right)$.

$c_4 > 0$ ~~…~~ less relevant
$c_4 < 0$ more relevant

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(3) Finally, $m^2(M)$ is sensitive to the new scale M:

$$m^2(M) = m_L^2(M) - \frac{M^2}{32\pi^2}\left(2\lambda_1^2 + \lambda_2\right) + \dots$$

This can destabilize the hierarchy $m_L^2 \ll M^2$
Even if we adjust $m_L^2(M) \ll M^2$, the corrections can destroy such hierarchy.
For instance take $(\lambda_2 + 2\lambda_1^2) < 0$ and $O(1)$ ....
This is the modern way to discuss the hierarchy problem.

We examine now the relation

$$C_4 = \lambda_0 - 3\lambda_1^2 - 4\lambda_1^2 \frac{m_L^2}{M^2}$$

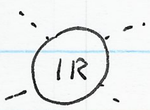
in the limit $\lambda_1 = 0$, and we look for the 1-loop corrections:

$$C_4 = \lambda_0 + \underset{\uparrow\, O(\hbar)}{\delta C}$$

Remark: in this limit $C_6 = 0$ at TL.

We evaluate the 4-point function:

<u>IR side</u>



$$= -i\, C_4 \cancel{\mu^{\epsilon}}$$

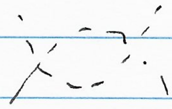
$$-\, i \frac{C_4^2}{32\pi^2} \cancel{\mu^{\epsilon}} \int_0^1 dx \left[\log \frac{m^2 - s\,x(1-x)}{\mu^2} + \text{"}t\text{"} + \text{"}u\text{"} \right]$$

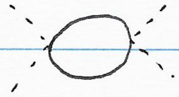
$+$ 

C_6 $\nearrow$ this gives $\underset{\uparrow}{C_6} \times O(\hbar)$

here we can use $(C_6)_{TL} = 0$

UV side

 $=$  $+$  + crossed

$+$  + crossed

$$= -i\lambda_0 - i\frac{\lambda_0^2}{32\pi^2}\int_0^1 dx\left[\log\frac{m_L^2 - s\,x(1-x)}{\mu^2} + \text{"}t\text{"} + \text{"}u\text{"}\right]$$

$$- i\frac{\lambda_2^2}{32\pi^2}\int_0^1 dx\left[\log\frac{M^2 - s\,x(1-x)}{\mu^2} + \text{"}t\text{"} + \text{"}u\text{"}\right]$$

$$= -i\lambda_0 - \dots$$

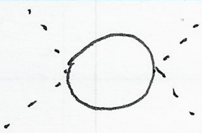
$$- i\frac{\lambda_2^2}{32\pi^2}\left[3\log\frac{M^2}{\mu^2} - \frac{\overbrace{s+t+u}^{4m^2 \text{ on-shell}}}{6M^2} + \dots\right]$$

Then equating the 4-point function evaluated on-shell we get:

$$-C_4 - \cancel{\frac{C_0^2}{32\pi^2}\int_0^1 dx\left[\log\frac{m^2 - s\,x(1-x)}{\mu^2} + \text{"}t\text{"} + \text{"}u\text{"}\right]} =$$

$$= -\lambda_0 - \cancel{\frac{\lambda_0^2}{32\pi^2}\int_0^1 dx\left[\log\frac{m_L^2 - s\,x(1-x)}{\mu^2} + \text{"}t\text{"} + \text{"}u\text{"}\right]}$$

$$- \frac{\lambda_2^2}{32\pi^2}\left[3\log\frac{M^2}{\mu^2} - \frac{2}{3}\frac{m^2}{M^2} + \dots\right]$$

 Symmetry factor $\frac{1}{4}\frac{1}{4}\,2\,2\,2 = \frac{1}{2}$ as 

$$\rightarrow \boxed{-C_4 = -\lambda_0 - \frac{3\lambda_2^2}{32\pi^2}\log\frac{M^2}{\mu^2} + \frac{\lambda_2^2}{48\pi^2}\frac{m^2}{M^2} + \dots}$$

choosing $\mu = M$

$$\boxed{C_4^{(M)} = \lambda_0^{(M)} - \frac{\lambda_2^2}{48\pi^2}\cdot\frac{m^2}{M^2} + \dots}$$

Moreover, as seen last lecture:

$$\mu\frac{\partial}{\partial\mu}C_4(\mu) = \frac{3C_4}{16\pi^2} + \dots$$

$$C_4(\mu) = \frac{C_4(M)}{1 - \frac{3C_4(M)}{16\pi^2}\log\frac{\mu}{M}}$$