Anomalies in QFT

We have encountered several ~~ex~~ examples of symmetries

→ Discrete (like P, T, CP)

→ <u>Continuous global</u> like the baryon number B the strangeness S, the (strong) hypercharge Y the chiral symmetries

$SU(3)_L \times SU(3)_R$ 3 flavors QCD $M=0$

$SU(2)_L \times SU(2)_R$ 2 flavors QCD $M=0$

→ <u>Continuous local</u> like

$U(1)_{em}$

$SU(2)_L \times U(1)_Y$ e.w.

$SU(3)_C$ QCD

Focus on the continuous ones.

Classically, to each generator T^a of a continuous symmetry, there is an associated conserved current via the Noether theorem:

$$\partial^\mu j_\mu^a(x) = 0 \qquad (1)$$

Such conservation law is <u>crucial</u> when dealing with <u>local</u> symmetries. If (1) fails, the theory cannot be constructed.

Think to QED. The Fock space (i.e. built with a_μ^+ acting on $|0\rangle$) has an indefinite metric an is not an Hilbert space.

To build the Hilbert space ~~th~~ we use the Gupta-Bleuler condition (G-B):

$$\partial_\mu A^{\mu(-)} |phys\rangle = 0 \quad \leftrightarrow \quad |phys\rangle \in \mathcal{H}$$

$\partial^\mu A_\mu^{(-)}$ denoting the negative frequency part i.e. the one containing annihilation operators

$$A_\mu(x) \approx a_\mu e^{i(kx-\omega t)} + a_\mu^+ e^{-i(kx+\omega t)}$$

The separation:

$$\partial^\mu A_\mu = \partial^\mu A_\mu^{(+)} + \partial^\mu A_\mu^{(-)}$$

is obvious in the free-case since $\Box A_\mu = 0$
$\rightarrow \Box \partial^\mu A_\mu = 0$
In the interacting case we have

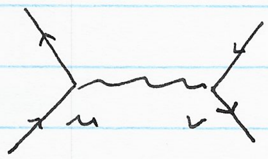
$$\Box A^\mu = e j^\mu_{em}$$

and $\Box \partial^\mu A_\mu = 0$ follows from $\partial_\mu j^\mu_{em} = 0$
similar considerations hold in the non-abelian case.

Another way of saying this is that $\partial_\mu j^\mu_{em} = 0$ is essential to show the independence of the scattering amplitudes on the gauge parameter ξ
consider the propagator:

$$\underset{\mu \qquad \nu}{\sim\!\sim\!\sim} = \frac{-i}{k^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - \frac{i}{\xi k^2}\frac{k_\mu k_\nu}{k^2} \equiv \Delta_{\mu\nu}$$

When computing, i.e. the amplitude



$$\approx J^\mu \Delta_{\mu\nu} J^\nu$$

$J^\mu \propto e\, \bar{v}(p)\gamma^\mu u(q)$ $\qquad p+q=k$

The ξ-dependent part of the amplitude is produced by the $\frac{k_\mu k_\nu}{k^2}$ term of the propagator leading to

$$e\, k_\mu \bar{v}(p)\gamma^\mu u(q) = e\, \bar{v}(p)(\not{p}+\not{q})\, u(q) \equiv 0$$

$$\Updownarrow$$

$$\partial_\mu j^\mu_{em} = 0$$

In the <u>global</u> case $\partial^\mu j_\mu^a \neq 0$ does not lead to inconsistencies.

An anomaly is a violation of the classical conservation law

$$\partial^\mu j_\mu^a = 0$$

due to quantum corrections.
We will be more precise in a moment.

$\rightarrow \partial^\mu j_\mu^a \neq 0 \qquad$ Anomaly

If T^a is the generator of a <u>global</u> symmetry, that's O.K. If the symmetry is <u>local</u> the theory is inconsistent.

How anomalies arise?

We have UV divergences that need a regularization procedure. A new parameter (e.g. $\varepsilon = 4-d$ in dimensional regularization) makes the amplitudes finite.
Divergences arise when sending ε to $\emptyset$.

In general the conservation law $\partial_\mu j^\mu_{cl} = 0$ gets modified into:

$$\partial_\mu j^\mu_{cl} = \varepsilon \underbrace{\Delta^0}_{\text{constant (i.e. } \varepsilon \text{ independent)}}$$

(still at the classical level)

Turning on quantum corrections $\Delta^0 \to \Delta(\varepsilon)$

$$\partial_\mu j^\mu = \varepsilon\, \Delta(\varepsilon)$$

$$= \varepsilon \left[\Delta^0 + \underbrace{\Delta^1(\varepsilon)}_{\text{1-loop}} + \underbrace{\Delta^2(\varepsilon)}_{\text{2-loops}} + \dots \right] \quad \text{etc...}$$

It may happen that

$$\Delta^1(\varepsilon) = \frac{\overbrace{\Delta^1_{res}}^{\text{constant} \neq 0}}{\varepsilon} + \underbrace{\Delta^1_{reg}(\varepsilon)}_{\text{finite in } \varepsilon \to 0 \text{ limit}}$$

As a result, in the limit $\varepsilon \to 0$:

$$\partial_\mu j^\mu = \Delta^1_{res} + \overbrace{\dots}^{\text{higher orders}}$$

we can still try to add to $\mathcal{L}_{sym}$ a counterterm $\delta\mathcal{L}$ whose contribution cancels Δ^1_{res}.
If this is not possible we have an anomaly.

The freedom to add (local) counterterms to the initial theory shows that Δ^1_{res} is not unambiguously defined.

*

Example:
axial symmetry of massless QED

$$\mathcal{L} = i\bar{\Psi}(\gamma^\mu\partial_\mu + ie\gamma^\mu A_\mu)\Psi - m\bar{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

invariant under
$U(1)_{em}$ local:

$$\begin{cases} \Psi(x) \to e^{-i\alpha(x)}\Psi(x) \\ \bar{\Psi}(x) \to e^{+i\alpha(x)}\bar{\Psi}(x) \\ A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x) \end{cases}$$

consider now
$U(1)_A$ global
β real
x-independent

$$\begin{cases} \Psi(x) \to e^{i\gamma_5\beta}\Psi(x) \\ \bar{\Psi}(x) \to \bar{\Psi}(x)e^{+i\beta\gamma_5} \end{cases}$$

infinitesimal form:

$$\delta\Psi(x) = i\beta\gamma_5\Psi(x)$$
$$\delta\bar{\Psi}(x) = +i\beta\bar{\Psi}(x)\gamma_5$$

since $\{\gamma^0, \gamma_5\} = 0$

$$\delta\bar{\Psi} = \delta(\Psi^+\gamma^0) = (\delta\Psi^+)\gamma^0 = -i\beta\Psi^+\gamma_5\gamma^0$$
$$= +i\beta\bar{\Psi}\gamma_5$$

The kinetic term is invariant since:

$$\delta(\bar\Psi \gamma^\mu \Psi) = \delta\bar\Psi \cdot \gamma^\mu \Psi + \bar\Psi \gamma^\mu \delta\Psi$$
$$= i\beta\left(\bar\Psi \gamma_5 \gamma^\mu \Psi + \bar\Psi \gamma^\mu \gamma_5 \Psi\right) = 0$$

The mass term not:

$$\delta \bar\Psi\Psi = \delta\bar\Psi \cdot \Psi + \bar\Psi \delta\Psi = 2i\beta\, \bar\Psi \gamma_5 \Psi$$

$$\rightarrow \quad \delta\mathcal{L} = -2i\beta m \bar\Psi \gamma_5 \Psi$$

$\delta\mathcal{L} = 0$ when $m = 0$. In this case the Noether current is

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu\Psi}\delta\Psi = i\bar\Psi\gamma^\mu\, i\beta\gamma_5\Psi$$
$$= -\beta\, \bar\Psi\gamma^\mu\gamma_5\Psi$$

We can take $j^\mu_A = \bar\Psi\gamma^\mu\gamma_5\Psi$

classically

$$\partial_\mu j^\mu_A = 0$$

Indeed

$$\partial_\mu j^\mu{}_A = \partial_\mu \bar\Psi\gamma^\mu\gamma_5\Psi$$
$$= \bar\Psi \overleftarrow{\partial\!\!\!/} \gamma_5 \Psi + \bar\Psi \overrightarrow{\partial\!\!\!/} \gamma_5 \Psi$$

e.o.m. $\quad i\partial\!\!\!/\Psi - m\Psi = 0$

$$-i\Psi^+ \overleftarrow{\partial\!\!\!/}^+ - \Psi^+ m = 0$$
$$-i\Psi^+ \overleftarrow{\partial\!\!\!/}^+ \gamma^0 - \bar\Psi m = 0$$
$$-i\bar\Psi\gamma^0 \overleftarrow{\partial\!\!\!/}^+ \gamma^0 - \bar\Psi m = 0 \quad \rightarrow \quad -i\bar\Psi\overleftarrow{\partial\!\!\!/} - \bar\Psi m = 0$$

$$\gamma^0\gamma^{\mu +}\gamma^0 = \gamma^\mu$$

$$\not{D}\psi = (\not{\partial} + ie\not{A})\psi \quad = \not{\partial}\psi + ie\not{A}\psi$$
$$\bar{\psi}\overleftarrow{\not{D}} = \bar{\psi}(\overleftarrow{\not{\partial}} - ie\not{A}) \quad = \bar{\psi}\not{\partial} - ie\bar{\psi}\not{A}$$

$$i\not{\partial}\psi = +e\not{A}\psi + m\psi$$
$$-i\bar{\psi}\overleftarrow{\not{\partial}} = +e\bar{\psi}\not{A} + \bar{\psi}m$$

$$\not{\partial}\psi = -ie\not{A}\psi - im\psi$$
$$\bar{\psi}\overleftarrow{\not{\partial}} = ie\bar{\psi}\not{A} + i\bar{\psi}m$$

$$\begin{aligned}\partial_\mu j^\mu{}_A = & (ie\bar{\psi}\not{A} + i\bar{\psi}m)\gamma_5\psi \\ & + \bar{\psi}(-\gamma_5)(-ie\not{A}\psi - im\psi) \\ = & \; ie\,\bar{\psi}(\not{A}\gamma_5 + \gamma_5\not{A})\psi \\ & + 2im\,\bar{\psi}\gamma_5\psi\end{aligned}$$

$$\boxed{\partial_\mu j^\mu{}_A = 2im\,\bar{\psi}\gamma_5\psi}$$

Including 1-loop correction we can show that:

$$\partial_\mu j^\mu{}_A = 2im\,\bar{\psi}\gamma_5\psi + \underbrace{\frac{e^2}{16\pi^2}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}}_{\text{anomaly}}$$

Can we add to $\mathcal{L}_{QCD}$ a counterterm to cancel this?

$$\begin{aligned}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} & = 2\epsilon_{\mu\nu\rho\sigma}\partial^\mu A^\nu \cdot F^{\rho\sigma} \\ & = 2\partial^\mu(\epsilon_{\mu\nu\rho\sigma}A^\nu F^{\rho\sigma}) \\ & \quad - 2\epsilon_{\mu\nu\rho\sigma}A^\nu\partial^\mu F^{\rho\sigma}\end{aligned}$$

Bianchi identity $\quad \epsilon_{\mu\nu\rho\sigma}\partial^\mu F^{\rho\sigma} \equiv 0$

$$\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} = 2\, \partial^\mu (\epsilon_{\mu\nu\rho\sigma} A^\nu F^{\rho\sigma})$$

No, we cannot.

General results:
For a gauge theory with gauge group G_{loc} and generators T^a, Noether currents satisfy

$$\partial^\mu j_\mu^a \propto tr(T^a \{T^b, T^c\})\, \epsilon_{\mu\nu\rho\sigma} F^{b\mu\nu} F^{c\rho\sigma}$$

→ where T^a are the generators associated to fermions in a left-handed basis

$$\psi_R \to C \overline{\psi}_R^{\,T} = (\psi^c)_L$$

→ $F_a^{\mu\nu}$ is the field strength

For a gauge theory with gauge group G and generators T^a satisfying a classical global symmetry under a group G_{gl} with generators T_{gl}^a we have

$$\partial^\mu j^a_{\mu\, gl} \propto tr(T^a_{gl} \{T^b, T^c\})\, \epsilon_{\mu\nu\rho\sigma} F_b^{\mu\nu} F_c^{\rho\sigma}$$

Apply this to QCD with <u>2 massless flavours</u> u, d where we turn on also e.m. interactions

$$G = SU(3)_c \times U(1)_{em}$$

$$G_{gl} = (e \to 0) = SU(2)_L \times SU(2)_R \times U(1)_B \times U(1)_A$$

1. What is the residual global symmetry when we turn on the e.m. interactions?
e.m. int distinguish u and d which, apparently, breaks the chiral symmetry $SU(2)_L \times SU(2)_R$

$$q_L \to \Omega_L q_L \qquad \Omega_L = e^{-i\alpha_L^a \frac{\sigma^a}{2}}$$
$$q_R \to \Omega_R q_R \qquad \Omega_R = e^{-i\alpha_R^a \frac{\sigma^a}{2}}$$

$$\mathcal{L}_{QCD+QED} = i\bar{q}\gamma^\mu D_\mu q - \frac{1}{4} G_a^{\mu\nu} G_{a\mu\nu}$$

$$D_\mu q = \left[\partial_\mu + ig_s t^a G_{a\mu} + ieQA_\mu\right]$$

$(t^a)_{\alpha\beta}$ acts on the color indices

$$Q = \begin{pmatrix} +2/3 & 0 \\ 0 & -1/3 \end{pmatrix}$$

$$\mathcal{L} = i\bar{q}\gamma^\mu\partial_\mu q - g_s\bar{q}\gamma^\mu t^a G_{a\mu} q$$
$$- eA_\mu \bar{q}\gamma^\mu Q q - \frac{1}{4} G_a^{\mu\nu} G_{a\mu\nu}$$

The 1st line is invariant under $SU(2)_L \times SU(2)_R$
2nd line

$$\delta\, A_\mu \bar{q} \gamma^\mu Q q =$$

$$A_\mu [\delta \bar{q} \gamma^\mu Q q + \bar{q} \gamma^\mu Q \delta q]$$

$$= A_\mu [\delta \bar{q}_L \gamma^\mu Q q_L + \bar{q}_L \gamma^\mu Q \delta q_L \quad + \quad L \to R]$$

$$= A_\mu (i \alpha_L^a) \left[\bar{q}_L \frac{\sigma^a}{2} \gamma^\mu Q q_L - \bar{q}_L \gamma^\mu Q \frac{\sigma^a}{2} q_L\right] + L \to R$$

$$= i \alpha_L^a A_\mu \bar{q}_L \gamma^\mu \left(\left[\frac{\sigma^a}{2}, Q\right]\right) q_L \quad \to \quad L \to R$$

This term is invariant (classically) only for the generators that commute with Q:

$$\frac{\sigma^3}{2}$$

Therefore the e.m. interactions explicitly break $SU(2)_L \times SU(2)_R \times U(1)_B \times U(1)_A$ into

$$U(1)_{3L} \times U(1)_{3R} \times U(1)_B \times U(1)_A$$

$$\begin{cases} U(1)_{3L} & q_L \to e^{-i\alpha_L^3 \frac{\sigma^3}{2}} q_L \\ U(1)_{3R} & q_R \to e^{-i\alpha_R^3 \frac{\sigma^3}{2}} q_R \end{cases}$$

Are these $U(1)$'s anomalous or not?

Exercise: compute the classically conserved Noether currents

$$j^\mu_{3L} = \bar{q}_L \gamma^\mu \frac{\sigma^3}{2} q_L$$

$$j^\mu_{3R} = \bar{q}_R \gamma^\mu \frac{\sigma^3}{2} q_R$$

$$j^\mu_B = \bar{q} \gamma^\mu q = \bar{u} \gamma^\mu u + \bar{u} \gamma^\mu d$$

$$j^\mu_A = \bar{q} \gamma^\mu \gamma_5 q = \bar{u} \gamma^\mu \gamma_5 u + \bar{u} \gamma^\mu \gamma_5 d$$

It is useful to combine $j^\mu_{3L,R}$ into

$$j^\mu_{3V} = j^\mu_{3L} + j^\mu_{3R} = \bar{q} \gamma^\mu \frac{\sigma^3}{2} q = \frac{1}{2}(\bar{u} \gamma^\mu u - \bar{d} \gamma^\mu d)$$

$$j^\mu_{3A} = j^\mu_{3R} - j^\mu_{3L} = \bar{q} \gamma^\mu \gamma_5 \frac{\sigma^3}{2} q$$

$$= \frac{1}{2}(\bar{u} \gamma^\mu \gamma_5 u - \bar{d} \gamma^\mu \gamma_5 d)$$

Fermion generators in left handed basis

		u_L	u_R	d_L	d_R
B	diag	1	1	1	1
A	diag	−1	+1	−1	+1
3V	$\frac{1}{2}$ diag	1	1	−1	−1
3A	$\frac{1}{2}$ diag	−1	+1	+1	−1

now $u_R \to (u^c)_L$ $d_R \to (d^c)_L$ changes the sign of the charge

$$u_L \quad (u^c)_L \quad d_L \quad (d^c)_L$$

B diag$(1 \quad -1 \quad 1 \quad -1)$
A diag$(-1 \quad -1 \quad -1 \quad -1)$
$3V$ $\frac{1}{2}$ diag$(1 \quad -1 \quad -1 \quad +1)$

$3A$ $\frac{1}{2}$ diag$(-1 \quad -1 \quad +1 \quad +1)$

There are no anomalies with B and $3V$ currents
For instance:

$$\mathrm{tr}\,(B \underbrace{\{t^a_c, t^b_c\}}_{\propto \mathbb{1} \text{ in flavour } u,d \text{ space}}) = 0$$

$$\mathrm{tr}\,(B\,\{Q, Q\}) = 2\,\mathrm{tr}\,(B\,Q^2)$$
$$= \left[\left(\tfrac{2}{3}\right)^2 (1-1) + \left(\tfrac{1}{3}\right)^2 (1-1)\right] \times 2 = 0$$

similarly for $3V$.

Consider now $3A$

$$\mathrm{tr}\,(3A \underbrace{\{t^a_c, t^b_c\}}_{\propto \mathbb{1} \text{ in flavourspace}}) = 0$$

$$\mathrm{tr}(3A\{Q,Q\}) = 2\left[\left(\tfrac{2}{3}\right)^2 - \left(\tfrac{1}{3}\right)^2\right] \times \frac{1}{2} \times N_c$$
$$= 2\,\frac{1}{2} \neq 0$$

Finally ~~A~~

$$\mathrm{tr}\, A\{t_c^a, t_c^b\} \neq 0$$

$$\mathrm{tr}\, A\{Q, Q\} \neq 0$$

The axial current has anomalies with both the gluons and the e.m. field.

~~x~~

For future use:

$$\boxed{\begin{aligned}\partial_\mu j^{3A} &= \frac{1}{2}\partial_\mu(\bar{u}\gamma^\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma_5 d)\\ &= \frac{1}{2}\frac{e^2}{16\pi^2}\frac{N_c}{3}\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}\end{aligned}}$$