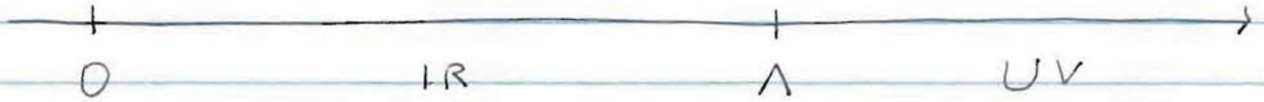
$\mathcal{L}_{IR}(\varphi_e, c')$ $\qquad\qquad$ $\mathcal{L}_{UV}(\varphi_e, \varphi_h, c)$



We derive Green functions for the theory in the UV from the functional

$$Z_{UV}[J_e, J_h] = \int \mathcal{D}\varphi_e \, \mathcal{D}\varphi_h \, e^{i\left[\underbrace{S_{UV}(\varphi_e, \varphi_h)}_{\text{defined in terms of } \mathcal{L}_{UV}(\varphi_e, \varphi_h, c)} + \int d^4x \, \varphi_e J_e + \int d^4x \cdot \varphi_h J_h\right]}$$

We are interested in

$$Z_{IR}[J_e] = \int d\varphi_e \, e^{i\left[\underbrace{S_{IR}(\varphi_e)}_{\mathcal{L}_{IR}(\varphi_e, c')} + \int d^4x \, J_e \varphi_e\right]}$$

In the IR region the 2 functionals have to make the same predictions.
This is automatically guaranteed if we define:

$$\boxed{e^{i S_{IR}(\varphi_e)} \equiv \int \mathcal{D}\varphi_h \, e^{i S_{UV}(\varphi_e, \varphi_h)}}$$

$S_{IR}(\varphi_e) \equiv S_W(\varphi_e)$ is called the Wilsonian action.

Indeed, in this case

$$Z_{IR}[J_e] = \int \mathcal{D}\varphi_e \int \mathcal{D}\varphi_h \, e^{i\left[S_{UV}(\varphi_e, \varphi_h) + \int d^4x \, J_e \varphi_e\right]}$$

$$= Z_{UV}[J_e, 0]$$

since in the IR we never have φ_h-particles in the initial or final state, $Z_{IR}[J_e]$ defined in this way gives rise to the same Green functions for φ_e as $Z_{UV}[J_e, J_h]$.

The task is that of determine $S_{IR}(\varphi_e)$.

Remarks:

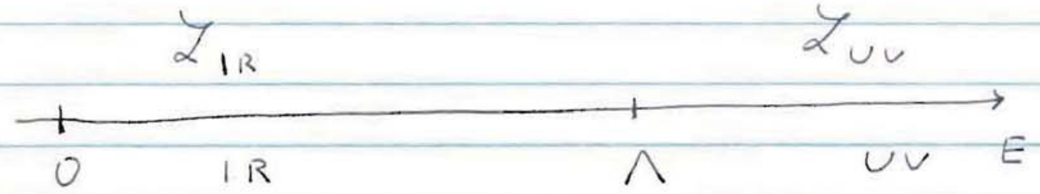
→ $S_{IR}(\varphi_e)$ is not yet something like $\int d^4x \, \mathcal{L}_{IR}(\varphi_e)$, since it is non local

→ We first start from a UV theory that remains always in the perturbative regime down to $E \ll \Lambda$.

→ We start by identifying $\mathcal{L}_{IR}(\varphi_e)$ at the tree-level

→ The identification of $\mathcal{L}_{IR}(\varphi_e)$ can be done order by order in perturbation theory.

In our specific problem of finding $\mathcal{L}_{IR}(\varphi_\ell, c')$ given $\mathcal{L}_{UV}(\varphi_h, \varphi_\ell, c)$:



The fundamental object is the Wilsonian action, defined through the path-integral:

$$e^{i S_W(\varphi_\ell)} \equiv \int \mathcal{D}\varphi_h \, e^{i S(\varphi_\ell, \varphi_h)}$$

If we have $S_W(\varphi_\ell)$ we can build all the functionals to compute the Green's functions of the theory. For instance

$$Z[j_\ell] \equiv \int \mathcal{D}\varphi_\ell \, e^{i\left[S_W(\varphi_\ell) + \int d^4x \, \varphi_\ell j_\ell\right]}$$

In general $S_W(\varphi_\ell)$ is a complicated non-local object, which we will express as a series of local operators when the expansion in powers of $(1/\Lambda)$ is performed.

This can be done order by order in perturb. theory if we are in the perturbative regime. We will start with a simple tree-level example:

$$\mathcal{L}_{UV} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}M^2\varphi^2 + \bar{\Psi}(i\partial\!\!\!/ - m)\Psi - g\varphi\bar{\Psi}\Psi$$

$\varphi \equiv \varphi_\ell$ $\qquad m \ll M \qquad$ φ real scalar

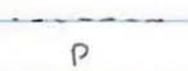
$\Psi \equiv \Psi_h$ $\qquad\qquad\qquad\quad$ Ψ Dirac fermion

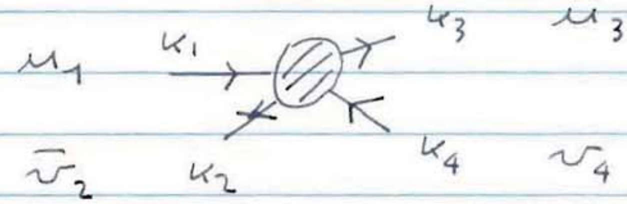
We have already an idea of the result since this is analogue to the Fermi limit of weak interactions.

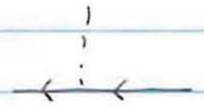
At $E \ll M$ we can only have Ψ states.

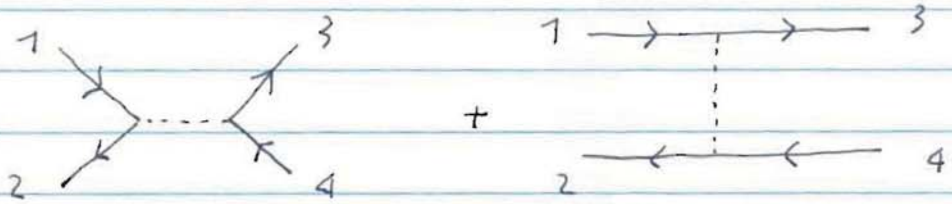
Consider the scattering $e^+ e^- \to e^+ e^-$

(call e^- the particle and e^+ the antiparticle)

 $= \dfrac{i}{p^2 - M^2}$



 $= -ig$



$s = (k_1 + k_2)^2$

$t = (k_1 - k_3)^2$

$$A(e^+e^- \to e^+e^-) = (-ig^2)\left[\frac{\bar{v}_2 u_1 \bar{u}_3 v_4}{s - M^2} + \frac{\bar{u}_3 u_1 \bar{v}_2 v_4}{t - M^2}\right]$$

$$\xrightarrow{s,\, t \ll M^2} \; = \frac{ig^2}{M^2}\left[\bar{v}_2 u_1 \bar{u}_3 v_4 + \bar{u}_3 u_1 \bar{v}_2 v_4\right]$$

which can be reproduced by:

$$\mathcal{L}_{IR}(\Psi) = \bar{\Psi}(i\not{\partial} - m)\Psi + \frac{g^2}{2M^2}(\bar{\Psi}\Psi\bar{\Psi}\Psi) + \dots$$

higher derivative terms

$$\boxed{S_W = \int d^4x\, \mathcal{L}_{IR}(\Psi) + \cdots}$$

But we would like to be more precise.

Integrating in a functional integral a set of fields φ_h <u>at the classical level</u> (= only tree-level diagrams) corresponds to solving the e.o.m for φ_h and substituting the solution in the original action. In formulae:
define:

$$e^{iS_W(\varphi_e)} = \int \mathcal{D}\varphi_h \, e^{iS(\varphi_e, \varphi_h)}$$

At the classical level we have

$$S_W(\varphi_e) = S(\varphi_e, \varphi_h^c(\varphi_e))$$

or

$$S_W(\varphi_e) = S(\varphi_e, \varphi_h^c(\varphi_e))(1 + O(\hbar))$$

where $\varphi_h^c(\varphi_e)$ is the solution of

$$\frac{\delta S(\varphi_e, \varphi_h)}{\delta \varphi_h} = 0$$

To see this we expand in the integrand $S(\varphi_e, \varphi_h)$ around $\varphi_h = \varphi_h^c(\varphi_e)$

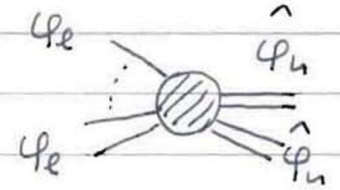
$$S(\varphi_e, \varphi_h) = S(\varphi_e, \varphi_h^c(\varphi_e)) + \frac{1}{2}\int d^4x_1 d^4x_2 \frac{\delta^2 S}{\varphi_h(x_1)\delta\varphi_h(x_2)}\Big| (\varphi_h(x_1) - \varphi_h^c(\varphi_e))(\varphi_h(x_2) - \varphi_h^c(\varphi_e)) + \dots$$

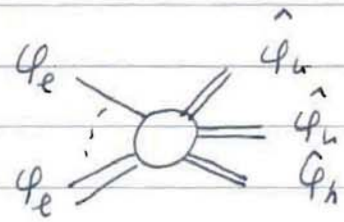
$$e^{iS_W(\varphi_e)} = e^{iS(\varphi_e, \varphi_h^c(\varphi_e))} \times \int \mathcal{D}\varphi_h \, e^{\frac{i}{2}\int d^4x_1 d^4x_2 \underbrace{\frac{\delta^2 S}{\varphi_h(x_1)\varphi_h(x_2)}}_{Q(\varphi_e)}\Big| \underbrace{(\varphi_h(x_1) - \varphi_h^c(\varphi_e))}_{\hat{\varphi}_h(x_1)}\overbrace{(\varphi_h(x_2) - \varphi_h^c(\varphi_e))}^{\hat{\varphi}_h(x_2)} + \dots}$$

change the integration variable $\mathcal{D}\hat{\varphi}_h \equiv \mathcal{D}\varphi_h$

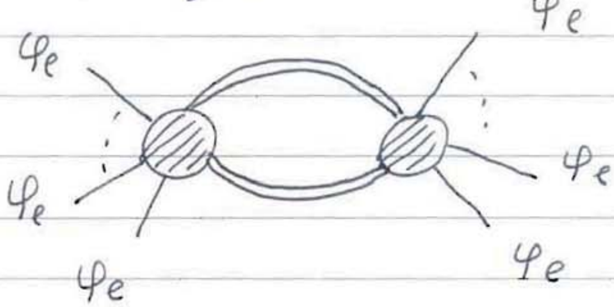
$$e^{iS_w(\varphi_e)} = e^{iS(\varphi_e,\varphi_h^c(\varphi_e))} \times \underbrace{\int \mathcal{D}\hat{\varphi}_h \, e^{\frac{i}{2}\int d^4x_1 d^4x_2 \, Q(\varphi_e)(x_1,x_2)\,\hat{\varphi}_h(x_1)\hat{\varphi}_h(x_2) + O(\hat{\varphi}_h^3)}}_{e^{iS_{loop}(\varphi_e)}}$$

All terms in the integrand are now $O(\hat{\varphi}_h^2)$ or higher. We can represent them as vertices of the type





They can contribute to $S_w(\varphi_e)$ only via loop diagrams, e.g



Therefore: $\quad S_w(\varphi_e) = S(\varphi_e, \varphi_h^c(\varphi_e))\big(1 + O(\hbar)\big)$

$\boxed{\text{Back to the top-down approach}}$

Start with $S_{UV}(\varphi_e, \varphi_h)$ at high energy
$m_\ell \equiv m \ll M \equiv m_h$ $\qquad E \ll M$
How $S_{IR} \equiv S_W(\varphi_e)$ looks like?
Which kind of operators contains?

$$e^{i S_{IR}(\varphi_e)} = \int \mathcal{D}\varphi_h \, e^{i S_{UV}(\varphi_e, \varphi_h)}$$

Consider the example of a Yukawa theory:

$$\mathcal{L}_{UV} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}M^2\varphi^2 + \bar{\Psi}(i\not{\partial} - m)\Psi - g\varphi\bar{\Psi}\Psi$$

→ φ 1 real scalar, Ψ Dirac fermion
→ Lorentz invariant
→ $M \gg m$

We use our general result, and determine $S_{IR}(\varphi_e)$ at tree-level

To find $\int \mathcal{D}\varphi$ by solving the e.o.m for φ:

$$\begin{cases} (\Box + M^2)\varphi = -g\bar{\Psi}\Psi \\ \text{boundary conditions} \equiv \text{b.c.} \end{cases}$$

We determine the propagator:

$$\begin{cases} (\Box_x + M^2)\Delta(x-y) = \delta^4(x-y) \\ \text{b.c.} \end{cases}$$

By working in momentum space

$$\delta^4(x-y) = \frac{1}{(2\pi)^4}\int d^4p \, e^{-ip(x-y)}$$

$$\Delta(x-y) = \frac{1}{(2\pi)^4}\int d^4p \, \tilde{\Delta}(p) \, e^{-ip(x-y)}$$

The equation becomes algebraic:

$$(-p^2+M^2)\,\tilde{\Delta}(p) = 1$$

$$\Delta(x-y) = \frac{-1}{(2\pi)^4}\int d^4p \, \frac{1}{p^2-M^2+i\varepsilon} \, e^{-ip(x-y)}$$

↳ from b.c.

The solution of the e.o.m for $\varphi(x)$ is

$$\varphi(x) = -g\int d^4y \, \Delta(x-y)\,(\bar{\Psi}\Psi)(y)$$

[check this by acting on $\varphi(x)$ with $(\Box_x + M^2)$]
Substituting this in $S_{UV}(\varphi,\Psi)$ we find:

$$\boxed{S_{IR}(\Psi) = \int d^4x \, \bar{\Psi}(i\not{\partial}-m)\Psi + \frac{g^2}{2}\int d^4x\,d^4y\,(\bar{\Psi}\Psi)(y) \times \Delta(x-y)(\bar{\Psi}\Psi)(x) + \text{loop corrections}}$$

$S_{IR}(\varphi_e)$ is, in general, a <u>non-local</u> functional [≡ fields evaluated at different space-time points]
However we can expand this in a series of local operators:

$$\Delta(x-y) = \frac{1}{(2\pi)^4}\int d^4p \, \frac{1}{M^2}\left(1+\frac{p^2}{M^2}+\frac{p^4}{M^4}+\dots\right)e^{-ip(x-y)}$$

$$= \frac{1}{M^2}\left(\delta^4(x-y) - \frac{\Box_x}{M^2}\delta^4(x-y) + \left(\frac{\Box_x^2}{M^4}\right)\delta^4(x-y)+\dots\right)$$

$$S_{IR}(\psi) = \int d^4x \Big[\underbrace{\bar\psi(i\not\partial - m)\psi}_{\text{relevant}} + \underbrace{\frac{g^2}{2M^2}\bar\psi\psi\bar\psi\psi}_{\text{leading irrelevant}} - \underbrace{\frac{g^2}{2M^4}\bar\psi\psi\Box\bar\psi\psi + \frac{g^2}{2M^6}\bar\psi\psi\Box^2\bar\psi\psi + \dots}_{\text{irrelevant}} \Big]$$

given any finite accuracy δ_{acc} (relative accuracy) in our computation we can truncate the series keeping the n-th term of the expansion such that.

$$\left(\frac{E^2}{M^2}\right)^n < \delta_{acc} \qquad E < M$$

> 1^{st} message:
> In general the top-down approach generates a non-local $S_{IR}(\varphi_e)$ which can be Taylor expanded in powers of some heavy scale keeping only a finite number of irrelevant operators in the EFT.

Also <u>relevant operators</u> can be generated in S_{IR}

Example: UV theory with 2 scalar fields φ_e and φ_h + symmetry $\begin{cases}\varphi_e \to -\varphi_e \\ \varphi_h \to \varphi_h\end{cases}$ $\begin{cases}\varphi_e \to \varphi_e \\ \varphi_h \to -\varphi_h\end{cases}$

$$\mathcal{L}_{UV} = \frac{1}{2}(\partial\varphi_e)^2 - \frac{1}{2}m^2\varphi_e^2 + \frac{1}{2}(\partial\varphi_h)^2 - \frac{\lambda}{4}(\varphi_h^2 - v^2)^2 - g^2\varphi_e^2\varphi_h^2$$

assuming the regime $\lambda v^2 \gg m^2$ and $\lambda \gg g^2$.

Treat g^2 term as a small perturbation.
First set $g^2 = 0$.

$$V(\varphi_h) = \frac{\lambda}{4}(\varphi_h^2 - v^2)^2$$

and φ_h has VEV v. Defining $\varphi_h = \hat{\varphi}_h + v$

$$V(\hat{\varphi}_h) = \frac{\lambda}{4}(\hat{\varphi}_h^2 + 2v\hat{\varphi}_h)^2$$

$$= \frac{\lambda}{4}\hat{\varphi}_h^4 + \lambda v \hat{\varphi}_h^3 + \lambda v^2 \hat{\varphi}_h^2$$

$\rightarrow$ $\hat{\varphi}_h$ has mass $M^2 = 2\lambda v^2 \gg m^2$
To derive $\mathcal{L}_{IR}$ we solve the eom for $\varphi_h (\leftrightarrow \hat{\varphi}_h)$ in the static (i.e. neglecting k. term) limit.

$$\varphi_h = v + O(g^2)$$

Plugging this into $\mathcal{L}_{UV}$ we find:

$$\mathcal{L}_{IR} = \frac{1}{2}(\partial\varphi_e)^2 - \frac{1}{2} m^2 \varphi_e^2 - g^2 v^2 \varphi_e^2 + \dots$$

relevant operator

$\rightarrow$ leads to a mass shift

$$\boxed{m_e^2 = m^2 + 2g^2v^2}$$

If v^2 very large, the effect can be huge.

Exercise: compute corrections of $O(g^2)$ to previous result.

$$V(\varphi_e, \varphi_n) = \frac{1}{2} m^2 \varphi_e^2 + \frac{\lambda}{4} (\varphi_n^2 - v^2)^2 + g^2 \varphi_e^2 \varphi_n^2$$

$$V_{\varphi_e} = m^2 \varphi_e + 2 g^2 \varphi_e \varphi_n^2 = 0$$
$$V_{\varphi_n} = \lambda \varphi_n (\varphi_n^2 - v^2) + 2 g^2 \varphi_e^2 \varphi_n = 0$$

solution

① $\varphi_e = 0 \quad \varphi_n = 0$

② $\varphi_e = 0 \quad \varphi_n = v$ minimum

$$V''\Big|_{min} = \begin{pmatrix} m^2 + 2g^2v^2 & 0 \\ 0 & 2\lambda v^2 \end{pmatrix}$$

assume $m^2 + 2g^2v^2 \ll 2\lambda v^2$

*

Eom in the static limit, keeping 1st correction

$$-\lambda \varphi_n (\varphi_n^2 - v^2) - 2g^2 \varphi_e^2 \varphi_n = 0$$

Solve as power series in g^2: $\varphi_n = v + c_1 g^2 + \dots$

$$\lambda (v + \cancel{c_1 g^2})(2 c_1 g^2 v + \cancel{O(g^4)}) + 2 g^2 \varphi_e^2 (v + \cancel{c_1 g^2}) = 0$$

$$\cancel{2} c_1 \lambda \cancel{g^2} v^2 + \cancel{2} \cancel{g^2} v \varphi_e^2 = 0$$

$$c_1 = -\frac{\varphi_e^2}{\lambda v}$$

$$\boxed{\varphi_n = v \left(1 - \frac{g^2}{\lambda v^2} \varphi_e^2 + \dots\right)}$$

Plug this into $\mathcal{L}_{UV}$:

$$\frac{\lambda}{4}(\varphi_h^2 - v^2)^2 + g^2\varphi_\ell^2\varphi_h^2 =$$

$$= \frac{\lambda}{4}\left[v^2\left(\cancel{1} - \frac{2g^2}{\lambda v^2}\varphi_\ell^2 + \frac{g^4}{\lambda^2 v^4}\varphi_\ell^4\right) - \cancel{v^2}\right]^2 +$$

$$+ g^2v^2\left(1 - \frac{2g^2}{\lambda v^2}\varphi_\ell^2 + \dots\right)\varphi_\ell^2$$

$$= + \frac{\cancel{\lambda}}{\cancel{4}}\,\frac{\cancel{4}g^4}{\lambda^{\cancel{2}}}\varphi_\ell^4 + \dots + g^2v^2\varphi_\ell^2 - \frac{2g^4}{\lambda}\varphi_\ell^4 + \dots$$

$$= g^2v^2\varphi_\ell^2 - \frac{g^4}{\lambda}\varphi_\ell^4$$

$$\Rightarrow$$

$$\boxed{\mathcal{L}_{IR} = \frac{1}{2}(\partial\varphi_\ell)^2 - \frac{1}{2}m^2\varphi_\ell^2 - \underset{\substack{\uparrow\\ \text{relevant}}}{g^2v^2\varphi_\ell^2} - \underset{\substack{\uparrow\\ \text{marginal}}}{\frac{g^4}{\lambda}\varphi_\ell^4} + \dots}$$

2^{nd} message:
The EFT obtained by Taylor expanding $S_{IR}(\varphi_\ell)$ in powers of some heavy scale and in powers of some small parameter contains a finite number of relevant and marginal operators and an infinite number of irrelevant operators (to practical purposes we can keep only a finite number of the latter)

Exercise

Estimate the parametric dependence of the φ_ℓ^6 operator

Notice: $\varphi_\ell^2 \leftrightarrow$  g^2v^2

$\varphi_\ell^4 \leftrightarrow$  $g^4\frac{1}{\lambda v^2}v^2 = g^4/\lambda$

$\varphi_\ell^6 \leftrightarrow$  $g^6\frac{1}{\lambda^2 v^4}v^2 = g^6/\lambda^2v^2$

$\varphi_\ell^{2n} \leftrightarrow \quad \dots\dots \quad = \dfrac{g^{2n}}{\lambda^{n-1}(v^2)^{n-2}}$

<u>Exercise</u>

Yukawa theory with $\lambda\varphi^4$ interaction:

$$\mathcal{L}_{UV} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}M^2\varphi^2 - \frac{\lambda}{4}\varphi^4 + \bar{\psi}(i\partial\!\!\!/ - m)\psi - g\varphi\bar{\psi}\psi$$

Derive $\mathcal{L}_{IR}$ at tree level keeping $O(\lambda)$ terms

$m \ll M$, neglecting the kinetic term

EOM in the static limit:

$$M^2\varphi + \lambda\varphi^3 + g\bar{\psi}\psi = 0$$

0^{th} order in λ:

$$\varphi_0 = -\frac{g\bar{\psi}\psi}{M^2} = \frac{J}{M^2} \qquad J \equiv -g\bar{\psi}\psi$$

1^{st} order:

$$\varphi_1 = \varphi_0 + \lambda\delta\varphi$$

$$\cancel{M^2\varphi_0} + \lambda M^2\delta\varphi + \lambda\varphi_0^3 \cancel{-J} = 0 \qquad \delta\varphi = -\frac{\varphi_0^3}{M^2}$$

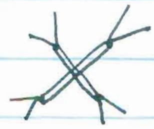
$$\boxed{\varphi_1 = \frac{J}{M^2}\left(1 - \lambda\frac{J^2}{M^6} + \dots\right)} \qquad \delta\varphi = -\frac{J^3}{M^8}$$

Back into $\mathcal{L}_{UV}$, neglecting the k. term

$$\mathcal{L}_{IR} = -\frac{1}{2}M^2\frac{J^2}{M^4}\left(1 - 2\lambda\frac{J^2}{M^6} + \dots\right) - \frac{\lambda}{4}\frac{J^4}{M^8} + \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

$$\underbrace{-g\bar{\psi}\psi}_{+J}\cdot\frac{J}{M^2}\left(1 - \lambda\frac{J^2}{M^6} + \dots\right)$$

$$= \frac{J^2}{M^2}\left(-\frac{1}{2} + 1\right) + \lambda\frac{J^4}{M^8}\left(+\cancel{1} - \frac{1}{4} - \cancel{1}\right) + \dots$$



$$= \frac{1}{2}\frac{J^2}{M^2} - \frac{\lambda}{4}\frac{J^4}{M^8} + \dots = \frac{1}{2}\frac{g^2(\bar{\psi}\psi)^2}{M^2} - \frac{\lambda g^4(\bar{\psi}\psi)^4}{4M^8} + \dots$$

 $g^6(\bar{\psi}\psi)^6$

Exercise

Spontaneously broken U(1) theory

φ complex scalar $\quad Q(\varphi) = +1$

ψ Dirac fermion $\quad Q(\psi) = +1$

$$\mathcal{L}_{UV} = \bar{\psi}(i\not{D} - m)\psi + (D_\mu\varphi)^\dagger D^\mu\varphi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{4}(|\varphi|^2 - v^2)^2$$

$$D_\mu\varphi = (\partial_\mu + igA_\mu)\varphi \qquad D_\mu\psi = (\partial_\mu + igA_\mu)\psi$$

Determine $\mathcal{L}_{IR}$ at tree-level in the static approximation in the regimes

1. $g^2v^2 \gg m^2, \lambda v^2$
2. $\lambda v^2 \gg m^2, g^2v^2$

Power counting for non-relativistic theories
Consider for instance:

$$\mathcal{L} = i \varphi^* \partial_t \varphi + \varphi^* \frac{\nabla^2}{2m} \varphi - c_2 \varphi^* \varphi - c_4 (\varphi^* \varphi)^2$$

$$\rightarrow \quad i \partial_t \varphi = -\frac{\nabla^2}{2m} \varphi + c_2 \varphi + \dots$$ Schroedinger-like equation

Determine dimensions for fields and operators.
Time and space cannot scale in the same way. Consider $d=4$ and

$$\begin{cases} t = e^{2\alpha} t' \\ \vec{x} = e^{\alpha} \vec{x}' \end{cases}$$

Now $d^4x = e^{5\alpha} d^4x'$ and the canonical dimension of φ is $3/2$.
We have

$5 > [O]$ relevant
$5 = [O]$ marginal
$5 < [O]$ irrelevant

$[\varphi^* \varphi] = 3$ relevant
$[(\varphi^* \varphi)^2] = 6$ irrelevant (This was marginal in relativistic QFT)

Relation to a relativistic real scalar φ_R is

$$\varphi_R = \frac{1}{2m} [e^{-imt} \varphi + e^{+imt} \varphi^*]$$

can be found in Kaplan [K]