SUMMARY

Anomalies of global classical symmetries in QCD + QED with 2 massless flavors

$$\mathcal{L} = i\,\bar{q}\,\gamma^\mu D_\mu q - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G_{a\mu\nu} G_a^{\mu\nu}$$

$$D_\mu q = \left[\partial_\mu + i g_s t^a G_{a\mu} + i e Q A_\mu\right] \begin{pmatrix} u \\ d \end{pmatrix} \qquad q \equiv \begin{pmatrix} u \\ d \end{pmatrix}$$

$$Q = \begin{bmatrix} +2/3 & 0 \\ 0 & -1/3 \end{bmatrix}$$

Classical global symmetries when $e = 0$

$$G_{ql}^{cl} = SU(2)_L \times SU(2)_R \times U(1)_B \times U(1)_A$$

$SU(2)_{L,R}$ $\quad q_{L,R} \to e^{-i \alpha^a_{L,R} \frac{\sigma^a}{2}} q_{L,R}$

$U(1)_B$ $\quad q \to e^{-i\frac{\alpha}{3}} q$

$U(1)_A$ $\quad q \to e^{-i\beta\gamma_5} q$

when $e \neq 0$ $\quad G_{ql}^{cl}(e=0)$ is explicitly broken down to

$$G_{ql}^{cl} = U(1)_{3L} \times U(1)_{3R} \times U(1)_B \times U(1)_A$$

$U(1)_{3L,R}$ $\quad q_{L,R} \to e^{-i\alpha^3_{L,R} \frac{\sigma^3}{2}} q_{L,R}$

[$\mathcal{L}$ has also discrete symmetries: C, CP and T.]

currents

$$j^\mu_{3L} = \bar{q}_L \gamma^\mu \frac{\sigma^3}{2} q_L \qquad j^\mu_{3R} = \bar{q}_R \gamma^\mu \frac{\sigma^3}{2} q_R$$

$$j^\mu_B = \frac{1}{3} \bar{q} \gamma^\mu q \qquad j^\mu_A = \bar{q} \gamma^\mu \gamma_5 q$$

$$j^\mu_{3V} = j^\mu_{3L} + j^\mu_{3R} = \bar{q} \gamma^\mu \frac{\sigma^3}{2} q$$

$$j^\mu_{3A} = - j^\mu_{3L} + j^\mu_{3R} = \bar{q} \gamma^\mu \gamma_5 \frac{\sigma^3}{2} q$$

$$= \frac{1}{2} (\bar{u} \gamma^\mu \gamma_5 u - \bar{d} \gamma^\mu \gamma_5 d)$$

Out of $j^\mu_{3V,A}$, j^μ_B, j^μ_A classically conserved 2 of them are anomalous :

$$\partial^\mu j_{\mu A} \neq 0$$ (anomalies with both $F_{\mu\nu}$ and $G_{a\mu\nu}$)

$$\partial^\mu j_{\mu 3A} = \frac{1}{2} \frac{e^2}{16\pi^2} \frac{N_c}{3} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

$$= A(x)$$

↑ e.m. tensor (no anomaly with $G_{a\mu\nu}$)

This result can be cast into an equivalent form:

Define:

$$e^{i\Gamma(A)} = \int \mathcal{D}q\,\mathcal{D}\bar{q}\; e^{i\int \mathcal{L}(q,\bar{q},A)\,d^4x} \qquad [\text{set } G_{a\mu}\equiv 0]$$

We can show that, under $q \to e^{-i\alpha\gamma_5\frac{\sigma^3}{2}} q$

$$\mathcal{D}q\,\mathcal{D}\bar{q} \to \mathcal{D}q\,\mathcal{D}\bar{q}\; e^{i\int d^4x\,\alpha\,A(x)}$$

Indeed, promoting α to a local parameter $\alpha(x)$ we have (change of variables):

$$0 = \delta\, e^{i\Gamma(A)} = \int \left[\delta(\mathcal{D}q\,\mathcal{D}\bar{q})\, e^{i\int d^4x\,\mathcal{L}} + \mathcal{D}q\,\mathcal{D}\bar{q}\; i\int d^4x\,\delta\mathcal{L}\, e^{\cdots}\right]$$

$$= \int \mathcal{D}q\,\mathcal{D}\bar{q}\;\, i\int d^4x \Big[\alpha(x)\,A(x) + \underbrace{\delta\mathcal{L}}_{\overset{!}{=}\,\partial_\mu\alpha\, j^\mu_{3A}}\Big]\, e^{i\int d^4x\,\mathcal{L}}$$

$$= \int \mathcal{D}q\,\mathcal{D}\bar{q}\;\, i\int d^4x\,\alpha(x)\left[\partial_\mu j^\mu_{3A} - A(x)\right] e^{i\int d^4x\,\mathcal{L}}$$

Therefore, under a ~~global~~ local transformation

$$\boxed{\left\langle \int d^4x\,\delta\mathcal{L} \right\rangle = \left\langle \int d^4x\,\alpha(x)\,A(x) \right\rangle}$$

Decay $\pi^0 \to \gamma\gamma$

We should add to our chiral Lagrangian

$$\mathcal{L}_{ch} = \frac{f^2}{4} \operatorname{tr}(\partial_\mu \Sigma^+ \partial^\mu \Sigma) + \frac{\langle \bar{q} q \rangle}{2} \operatorname{tr}(M\Sigma + h.c.)$$

$$\Sigma \equiv e^{\frac{2i\pi}{f}} \qquad \pi \equiv \frac{\sigma^a}{2}\pi^a$$

$$= \begin{bmatrix} \pi^0/2 & \pi^+/\sqrt{2} \\ \pi^-/\sqrt{2} & -\pi^0/2 \end{bmatrix}$$

$$M = \begin{bmatrix} m_u & 0 \\ 0 & m_d \end{bmatrix}$$

2 flavours u, d are sufficient

The electromagnetic interaction.
Observe that $\mathcal{L}_{ce}$, in the $M \to 0$ limit, is invariant under parity P, $SU(2)_L \times SU(2)_R$

em interactions can be introduced by asking invariance under local $U(1)_{em}$ transformations

$$\Sigma \to e^{-ie\alpha Q}\, \Sigma\, e^{+ie\alpha Q} \qquad Q = \begin{bmatrix} +2/3 & 0 \\ 0 & -1/3 \end{bmatrix}$$

$$\delta\Sigma = -ie\alpha[Q, \Sigma]$$

$$\cancel{[Q, \Sigma] = \begin{bmatrix} 2/3 & 0 \\ 0 & -1/3 \end{bmatrix}}$$

$$\delta(e^{\frac{2i\pi}{f}}) = 2i\frac{\delta\pi}{f} + \dots$$

$$\to \quad \pi \longrightarrow e^{-ie\alpha Q}\, \pi\, e^{+ie\alpha Q}$$

$$\delta\pi = -ie\alpha\,[Q,\pi]$$

$$\delta \begin{pmatrix} \pi^0/2 & \pi^+/\sqrt{2} \\ \pi^-/\sqrt{2} & -\pi^0/2 \end{pmatrix} = -ie\alpha \left[\begin{pmatrix} 2/3\,\cancel{\pi^0/2} & 2/3\,\pi^+/\sqrt{2} \\ -1/3\,\pi^-/\sqrt{2} & -\cancel{\tfrac{1}{3}\pi^0/2} \end{pmatrix} - \begin{pmatrix} \cancel{2/3\,\pi^0/2} & -1/3\,\pi^+/\sqrt{2} \\ 2/3\,\pi^-/\sqrt{2} & -\cancel{1/3\,\pi^0/2} \end{pmatrix} \right]$$

$$= -ie\alpha \begin{pmatrix} 0 & \pi^+/\sqrt{2} \\ -\pi^-/\sqrt{2} & 0 \end{pmatrix}$$

$$\delta\pi^0 = 0 \qquad \delta\pi^\pm = \mp ie\alpha\,\pi^\pm \qquad \text{o.k.}$$

covariant derivative

$$D_\mu\Sigma = \partial_\mu\Sigma + ieA_\mu[Q,\Sigma]$$

check that under $U(1)_{em}$, $D_\mu\Sigma$ transform as

$$D_\mu\Sigma \xrightarrow{U(1)_{em}} e^{-ie\alpha Q}\, D_\mu\Sigma\, e^{ie\alpha Q}$$

The Lagrangian become:

$$\mathcal{L}_{ch} = \frac{f^2}{4}\,\mathrm{tr}\, D_\mu\Sigma^+ D^\mu\Sigma + \frac{\langle\bar{q}q\rangle}{2}\,\mathrm{tr}\,(M\Sigma + \text{h.c.})$$

Now, in the massless limit, $\mathcal{L}_{ch}$ is invariant under P and under $U(1)_{3L}\times U(1)_{3R}$

$$\Sigma \to e^{-i\alpha_L^3 \frac{\sigma^3}{2}} \, \Sigma \, e^{+i\alpha_R^3 \frac{\sigma^3}{2}}$$

We can also combine these 2 $U(1)$s into

$$U(1)_{3V}: \quad \Sigma \to e^{-i\alpha_V \frac{\sigma^3}{2}} \, e^{+i\alpha_V \frac{\sigma^3}{2}} \qquad (1)$$

i.e. we take $\alpha_L^3 = \alpha_R^3 \equiv \alpha_V$

$$U(1)_{3A}: \quad \Sigma \to e^{i\alpha_A \frac{\sigma^3}{2}} \, \Sigma \, e^{i\alpha_A \frac{\sigma^3}{2}} \qquad (2)$$

i.e. we take $\alpha_R^3 = -\alpha_L^3 \equiv \alpha_A$

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How π's transform under $U(1)_{3V,A}$?

(1) $$\delta\pi = -i\frac{\alpha_V}{2}[\sigma^3, \pi]$$

$$= -i\frac{\alpha_V}{2}\left[\begin{pmatrix} \cancel{\pi^0}/2 & \pi^+/\sqrt{2} \\ -\pi^-/\sqrt{2} & -\cancel{\pi^0}/2 \end{pmatrix} - \begin{pmatrix} \cancel{\pi^0}/2 & -\pi^+/\sqrt{2} \\ \pi^-/\sqrt{2} & -\cancel{\pi^0}/2 \end{pmatrix}\right]$$

$$= -i\alpha_V \begin{pmatrix} 0 & \pi^+/\sqrt{2} \\ -\pi^-/\sqrt{2} & 0 \end{pmatrix} \to \boxed{\begin{aligned} \delta\pi^0 &= 0 \\ \delta\pi^\pm &= \mp i\alpha_V \pi^\pm \end{aligned}} \quad (1)$$

(2)

$$\delta\left(1 + \frac{2i\pi}{f} + \dots\right) = \left(1 + i\alpha_A\frac{\sigma^3}{2}\right)\left(1 + \frac{2i\pi}{f} + \dots\right)\left(1 + i\alpha_A\frac{\sigma^3}{2}\right) - \left(1 + 2i\frac{\pi}{f} + \dots\right)$$

$$= i\alpha_A\sigma^3 - \frac{2\alpha_A}{2f}(\sigma^3\pi + \pi\sigma^3) + \dots$$

$$\frac{2i}{f}\delta\pi = 2i\alpha_A\frac{\sigma^3}{2} - \frac{\alpha_A}{f}\begin{pmatrix} 2\pi^0 & 0 \\ 0 & -2\pi^0 \end{pmatrix}$$

$$\delta\begin{bmatrix} \pi^0 & \pi^+/\sqrt{2} \\ \pi^-/\sqrt{2} & -\pi^0 \end{bmatrix} = \begin{bmatrix} f/2 & 0 \\ 0 & -f/2 \end{bmatrix}\alpha_A + i\alpha_A\begin{bmatrix} \pi^0/2 & 0 \\ 0 & -\pi^0/2 \end{bmatrix}$$

$$\delta\left(1\!\!1 + \frac{2i\pi}{f}\right) = \left(1 + i\alpha_A \frac{\sigma^3}{2} + \dots\right)\left(1\!\!1 + \frac{2i\pi}{f} + \dots\right)\left(1 + i\alpha_A \frac{\sigma^3}{2} + \dots\right)$$

$$- \left(1\!\!1 + 2i\frac{\pi}{f} + \dots\right)$$

$$= i\alpha_A \sigma^3 - \frac{\alpha_A}{f}(\sigma^3 \pi + \pi\sigma^3) + \dots$$

$$\delta\pi = \frac{\cancel{2}f}{\cancel{2}\cancel{i}} \cancel{i}\alpha_A \frac{\sigma^3}{2} - \frac{\cancel{f}}{2i}\frac{\alpha_A}{\cancel{f}}(\sigma^3\pi + \pi\sigma^3) + \dots$$

$$\delta\pi = \alpha_A f \frac{\sigma^3}{2} + i\alpha_A\left(\frac{\sigma^3}{2}\pi + \pi\frac{\sigma^3}{2}\right) + \dots$$

$$\frac{\sigma^a}{2}\delta\pi^a = \alpha_A f \frac{\sigma^3}{2} + \frac{i}{2}\alpha_A \begin{bmatrix} \pi^0 & 0 \\ 0 & -\pi^0 \end{bmatrix} + \dots$$

$$\to \boxed{\begin{aligned} &\delta\pi^\pm = 0 \\ &\delta\pi^0 = \alpha_A f + i\alpha_A \pi^0 + \dots \end{aligned}} \quad (2)$$

The e.m. invariant π Lagrangian does not allow π^0 to couple directly to photons

$$D_\mu \Sigma = \partial_\mu \Sigma + ie A_\mu [Q, \Sigma]$$

$$= \partial_\mu e^{\frac{i2\pi}{f}} + ie A_\mu \cdot \frac{2i}{f}[Q, \pi]$$

$$= \frac{2i}{f}\partial_\mu \pi + \frac{1}{2}\left(\frac{2i}{f}\right)^2 \partial_\mu \pi^2 + \dots - \frac{2e}{f} A_\mu \begin{bmatrix} 0 & \pi^+/\sqrt{2} \\ -\pi^-/\sqrt{2} & 0 \end{bmatrix}$$

$$+ ie\left(\frac{2i}{f}\right)^2 A_\mu \underbrace{\left[Q, \frac{\pi^{0\,2} + 2\pi^+\pi^-}{4}\cdot 1\!\!1\right]}_{0} + \dots$$

$$= \frac{2i}{f}\partial_\mu \pi - \frac{2}{f^2}\partial_\mu \pi^2 - \frac{2e}{f} A_\mu \begin{bmatrix} 0 & \pi^+/\sqrt{2} \\ -\pi^-/\sqrt{2} & 0 \end{bmatrix} + \dots$$

To describe the decay $\pi^0 \to \gamma\gamma$ we would need a term like

$$\frac{\pi^0}{\Lambda} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

→ gauge invariant: $F_{\mu\nu}$

→ P, CP invariant:

$$\pi^0 \to -\pi^0 \qquad F^{\mu\nu} \to -F_{\mu\nu}$$

$$\to \left(\frac{-\pi^0}{\Lambda}\right) \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = +\frac{\pi^0}{\Lambda} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

→ invariant under $U(1)_{3V}$ since $\delta\pi^0 = 0$

→ <u>NOT invariant</u> under $U(1)_{3A}$ since $\delta\pi^0 = \alpha_A f$

$$\delta\left[\frac{\pi^0}{\Lambda} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}\right] = \alpha_A \frac{f}{\Lambda} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

$$\not\propto$$

Back to QCD + QED (2 flavours)

$$U(1)_{3A} \quad \to \quad j^\mu_{3A} = \frac{1}{2}\left(\bar{u}\gamma^\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma_5 d\right)$$

Remember:

$$\partial_\mu j^\mu_{3A} = \mathcal{A} = \frac{1}{2} \frac{e^2}{16\pi^2} \frac{N_c}{3} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

$$\updownarrow$$

$$\boxed{\left\langle \int d^4x\, \delta\mathcal{L} \right\rangle = \left\langle \int d^4x\, \alpha(x)\, \mathcal{A}(x) \right\rangle}$$

For consistency we ask that the same transform. properties applies to $\mathcal{L}_{ch}$ as well:

$$\langle i\int d^4x\, \delta\mathcal{L}_{ch}\rangle = \int \mathcal{D}\pi\ \delta\, e^{i\int d^4x\, \mathcal{L}_{ch}}$$

$$= \int \mathcal{D}\pi\ i\int d^4x\, \alpha(x) A(x)\, e^{i\int d^4x\, \mathcal{L}_{ch}}$$

We see that we should add to the chiral Lagrangian a term $\mathcal{L}_A$ such that

$$\delta\mathcal{L}_A = \int d^4x\, \alpha(x)\, A(x)$$

Since $\delta\pi^0 = \alpha(x) f$, such a term is

$$\mathcal{L}_A = \int d^4x\, \frac{\pi^0(x)}{f} A(x)$$

Therefore the EFT action reproducing all transf. properties of $\mathcal{L}_{QCD}$ is

$$\mathcal{L}_{ch} = \frac{f^2}{4} \mathrm{tr}\, D_\mu\Sigma^+ D^\mu\Sigma + \frac{\langle\bar{q}q\rangle}{2} \mathrm{tr}\,(M\Sigma + \mathrm{h.c.})$$

$$+ \underbrace{\frac{\pi^0}{f}\frac{e^2}{32\pi^2}\frac{N_c}{3}}_{\text{cfr } \frac{\pi^0}{\Lambda}} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}$$

From $\mathcal{L}_{ch}$ we can compute $\Gamma(\pi^0\to\gamma\gamma) \propto \frac{\alpha^2}{f^2} N_c^2$

$$\Gamma(\pi^0\to\gamma\gamma) = \frac{\alpha^2}{4}\frac{m_\pi^3}{144\pi^3 f^2} N_c^2 = \left(\frac{N_c}{3}\right)^2 \times 1.11\ 10^{16}\ \mathrm{sec}^{-1}$$

cfr $\Gamma_{exp}(\pi^0\to\gamma\gamma) = (1.19 \pm 0.08)\ 10^{16}\ \mathrm{sec}^{-1}$

BR $(\pi^0\to\gamma\gamma) \simeq 99\%$ $\quad\to N_c = 3$