CCWZ Callan Coleman Wess Zumino (1969)
Phenomenological Lagrangian.

We have seen in 3 flavors massless QCD
π^a $(a = 1 \dots 8)$ describe a real manifold M
The global group $G = SU(3)_L \times SU(3)_R$ acts on M
through

$$e^{\frac{2i}{f}\pi} \equiv \Sigma \to \Omega_L \Sigma \Omega_R^+ \equiv \Sigma' \equiv e^{\frac{2i}{f}\pi'}$$

there is a special point on M : $\pi^a = 0$ (the origin) $\leftrightarrow$ $\Sigma = 1\!\!1$, left invariant by $H = SU(3)_V$

$$1\!\!1 \to \Omega \, 1\!\!1 \, \Omega^+ \equiv 1\!\!1 \qquad (\Omega_L = \Omega_R = \Omega)$$

This describes the breaking of $SU(3)_L \times SU(3)_R$ down to $SU(3)_V$.

We want to generalize this description to G (continuous, compact, connected semisimple) spontaneously broken down to H (continuous)

Consider the (real) manifold of scalar fields $\varphi(x) \in M$
and the general action of G on M

$$\varphi \to g\varphi \qquad \text{(in general non linear)}$$

Let M has a special point φ_0 left invariant by the subgroup H:

$$\varphi_0 \to h\varphi_0 \equiv \varphi_0$$

Not restrictive to choose $\varphi_0 = 0$, the origin.

1. In general, also the group H

$$\varphi \to h\varphi \qquad h \in H$$

has a non linear action on M.
However, it is always possible, by a <u>local field redefinition</u>, to find new coordinates $\tilde{\varphi}$, such that the action of H is linear,

$$h\tilde{\varphi} = D(h)\tilde{\varphi}$$

↳ linear representation of H

<u>Proof</u>
Let dh be an invariant measure on H, normalized as

$$\int_H dh = 1 \qquad d\, hh' \equiv dh$$

Expand $h\varphi$ in powers of φ around $\varphi_0 \equiv 0$:

$$h\varphi = \text{const} + D(h)\varphi + O(\varphi^2)$$

$$h\emptyset = \text{const} = \emptyset$$

$$h\varphi = D(h)\varphi + O(\varphi^2)$$

Define: $$\tilde{\varphi} = \int_H dh\, D^{-1}(h)\, h\varphi = \varphi + O(\varphi^2)$$

(local field redefinition)

$$h_0 \tilde{\varphi} = \int_H dh \, D^{-1}(h) \, h h_0 \varphi$$

$$= \int_H d(\underbrace{h h_0}) \, D^{-1}(\underbrace{h h_0} h_0^{-1}) \, \underbrace{h h_0} \varphi$$

$$= \int_H dh' \, D(h_0) \, D^{-1}(h') \, h' \varphi$$

$$= D(h_0) \, \tilde{\varphi}$$

$\to$ $\tilde{\varphi}$ transform linearly

$D(h)$ is a linea representation

$$h_1 \underbrace{h_2 \varphi}_{\varphi'} = D(h_1 h_2) \varphi + O(\varphi^2)$$

$$= h_1 \varphi' = D(h_1) \varphi' + O(\varphi'^2)$$

$$= D(h_1) (D(h_2) \varphi + O(\varphi^2)) + O(\varphi'^2))$$

$$= D(h_1) D(h_2) \varphi + O(\varphi^2)$$

$$\to \quad D(h_1 h_2) = D(h_1) D(h_2)$$

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Since $h \varphi_0 = \varphi_0$ $\quad h \in H$

but not for a generic G, we expect

d Goldstone bosons $\quad d = \dim(G/H)$

Parametrization of M

$\dim G = n$ $\qquad \dim H = n - d$

$\to \quad \dim G/H = d \quad \to \quad d$ Goldstone bosons

$\to \quad \dim M \geq d$

Let (V_i, A_l) $\qquad \begin{cases} i = 1 \dots n-d \\ l = 1 \dots d \end{cases}$
be an
orthonormal set of generators of G
such that $\begin{cases} V_i \text{ are generators of } H \\ A_l \text{ are generators in } G/H \end{cases}$

Each element $g_0 \in G$ admits the unique decomposition:

$$g_0 = e^{\xi_l A_l} \, e^{u_i V_i}$$
$$\equiv e^{\xi A} \, e^{u V}$$

The idea is to promote ξ_l to fields $\xi_l(x)$ providing (a subset of) parameters of M.
We should define the action of G on ξ_l.
We use, for any $g \in G$

$$g \, e^{\xi A} = e^{\xi^g(\xi) A} \, e^{u^g(\xi) V}$$

Thus we define:

$$\xi_l \xrightarrow{g \in G} \xi_l^g(\xi) \qquad (1)$$

as transformation of the coordinates ξ_l.

if dim $M > d$ we need other coordinates ψ. We can redefine these coordinates such that they transform linearly under H:

$$\psi \xrightarrow{h \in H} D(h)\,\psi$$

↳ linear realization

We extend this transformation to the whole G, by defining:

$$\psi \xrightarrow{g \in G} D(e^{u^g(\xi) V})\,\psi \qquad (2)$$

Notice that while (1) is self consistent, (2) requires the existence of the coordinate ξ and the action of G on them.

We now have a complete set of coordinates (ξ, ψ) on M that under G transform as

$$\begin{cases} \xi \longrightarrow \xi^g(\xi) \\ \psi \longrightarrow D(e^{u^g(\xi) V})\,\psi \end{cases} \qquad (3)$$

where (ξ^g, u^g) are defined by:

$$g\, e^{\xi A} = e^{\xi^g(\xi) A}\, e^{u^g(\xi) V}$$

The transformations (3) provide a (non-linear) realization of G.
To see this we have to show that:

$$\xi \xrightarrow{g_1} \xi^{g_1}(\xi) \xrightarrow{g_2} \xi^{g_1}(\xi^{g_2}(\xi)) \equiv \xi^{g_1 g_2}(\xi)$$

$$\psi \xrightarrow{g_1} D(e^{u^{g_1}(\xi)V})\psi \xrightarrow{g_2} D(e^{u^{g_1}(\xi^{g_2}(\xi))V}) \cdot D(e^{u^{g_2}(\xi)V})\psi$$

$$\equiv D(e^{u^{g_1 g_2}(\xi)V})\psi$$

$$g_2 e^{\xi A} = e^{\xi^{g_2}(\xi)A} e^{u^{g_2}(\xi)V}$$

$$g_1 g_2 e^{\xi A} = e^{\xi^{g_1 g_2}(\xi)A} e^{u^{g_1 g_2}(\xi)V}$$

$$= g_1 e^{\xi^{g_2}(\xi)A} e^{u^{g_2}(\xi)V}$$

$$= e^{\xi^{g_1}(\xi^{g_2}(\xi))A} e^{u^{g_1}(\xi^{g_2}(\xi))V} e^{u^{g_2}(\xi)V}$$

O.K.

The coordinates / parametrization (ξ, ψ) + (3) is called <u>standard</u>.

If we restrict to $h \in H$, we have

$$h e^{\xi A} = \underbrace{h e^{\xi A} h^{-1}}_{e^{\xi^h A}} \underbrace{h}_{e^{u^h V}}$$

the (3) becomes

$$\begin{cases} e^{\xi A} \to h e^{\xi A} h^{-1} \\ \psi \to D(h)\psi \end{cases}$$ <u>is linear</u>

Remark

In the standard parametrization, the point $(\xi, \psi) = 0$ is left invariant by the subgroup H:

$$\underbrace{e^{0A}}_{1\!1} \to h \underbrace{e^{0A}}_{1\!1} h^{-1}$$

and $D(h)\psi = 0$ if $\psi = 0$

If we act with a generic g out of H

$$g = e^{\alpha A}$$

$$g\, e^{\xi A} = e^{\alpha A} e^{\xi A} = e^{(\xi + \alpha) A} \quad \ldots$$

$$\xi \to \xi + \alpha$$

and the origin is no more invariant

→ G is spontaneously broken down to H

→ ξ describe the Goldstone bosons, since $\xi \to \xi + \alpha + \ldots$ $g = e^{\alpha A}$

Main results (not proved here):
By a local field redefinition, any (non-linear) realization of G on M can be put into the standard form.

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How to build an invariant Lagrangian

Consider 2 cases

① Symmetric coset G/H

In general

$$[V_i, V_j] = i f_{ijk} V_k$$
$$[V_i, A_l] = i f_{ilm} A_m$$
$$[A_l, A_m] = i f_{lmn} A_n + i f_{lmk} V_k$$

if $f_{lmn} = 0$ the coset is symmetric and the above algebra is invariant under the reflection R

$$\begin{cases} V_i \to V_i \\ A_l \to -A_l \end{cases}$$

We have $\quad g\, e^{\xi A} = e^{\xi^g A} e^{u^g V}$

and

$$R(g)\, e^{-\xi A} = e^{-\xi^g A} e^{u^g V}$$

$$e^{\xi A} R(g^{-1}) = e^{-u^g V} e^{+\xi^g A}$$

$$\to \boxed{g\, e^{2\xi A} R(g^{-1}) = e^{2\xi^g A}}$$

See exampee:

~~For symmetric coset $e^{2\xi A}$ transform linearly under G and we can apply the known rules to build an invariant Lagrangian, depending on ξ.~~

2 Non symmetric coset

We cannot write invariants depending on ξ with no derivatives

$$\mathcal{L}(\xi) = \mathcal{L}(\xi^g)$$

if we take $g = e^{-\xi A}$ $\qquad \xi^g = 0$

$\rightarrow \quad \mathcal{L}(\xi) = \mathcal{L}(0) = \text{constant}$

this applies to case 1 as well.

$\mathcal{L}$ should depend on $\partial_\mu \xi$, ψ, $\partial_\mu \psi$

we need quantities with good transf. properties. Consider

$$e^{-\xi A} \partial_\mu e^{\xi A} = (p_\mu A + v_\mu V) \in G \text{ lie algebra}$$

$$\Big\downarrow g$$

$$e^{-\xi^g A} \partial_\mu e^{\xi^g A} = e^{u^g V} e^{-\xi A} \cancel{g^{-1}} \partial_\mu \left(\cancel{g} e^{\xi A} e^{-u^g V} \right)$$

$$= e^{u^g V} \underbrace{\left(e^{-\xi A} \partial_\mu e^{+\xi A} \right)}_{p_\mu A + v_\mu V} e^{-u^g V} + e^{u^g V} \partial_\mu e^{-u^g V}$$

$$\rightarrow \begin{cases} p'_\mu A = e^{u^g V} p_\mu A \, e^{-u^g V} \\ v'_\mu V = e^{u^g V} v_\mu V e^{-u^g V} - (\partial_\mu e^{u^g V}) e^{-u^g V} \end{cases}$$

$v_\mu V$ can be used to build a covariant derivative

$$D_\mu \Psi = (\partial_\mu + v_\mu V)\Psi$$

$$\to \partial_\mu (e^{u^g V} \Psi) + e^{u^g V} v_\mu V \cancel{e^{-u^g V}} \cancel{e^{u^g V}} \Psi$$

$$- (\partial_\mu e^{u^g V}) \cancel{e^{-u^g V}} \cancel{e^{u^g V}} \Psi$$

$$= e^{u^g V} (\partial_\mu + v_\mu V) \Psi = e^{u^g V} \Psi$$

We have 3 objects transforming with $e^{u^g V}$:

$$\begin{cases} \Psi \to e^{u^g V} \Psi \\ D_\mu \Psi \to e^{u^g V} D_\mu \Psi \\ p_\mu A \to e^{u^g V} p_\mu A \, e^{-u^g V} \end{cases}$$

When we restrict to H, $e^{u^g V} \equiv h$
Therefore, if we build

$\mathcal{L}(\Psi, D_\mu \Psi, p_\mu)$ invariant under $h \in H$
then it is automatically invariant under the whole G.

Example

$G = SU(2)_L \times SU(2)_R$ $\qquad H = SU(2)_V$

the coset is symmetric

generators L_i, R_i

$$[L_i, L_j] = i\epsilon_{ijk} L_k$$
$$[R_i, R_k] = i\epsilon_{ike} R_e$$
$$[L_i, R_j] = 0$$

realized e.g. by

$$L_i = \begin{pmatrix} \frac{\sigma^i}{2} & 0 \\ 0 & 0 \end{pmatrix} \qquad R_i = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sigma^i}{2} \end{pmatrix}$$

Define $\quad V_i = L_i + R_i$
$\qquad A_i = L_i - R_i$

exercise: find the commutators and check that the coset $SU(2)_L \times SU(2)_R / SU(2)_V$ is symmetric

$$\rightarrow \quad e^{2i\xi A} \longrightarrow g\, e^{2i\xi A} R(g^{-1})$$

in particular, if we take $g = e^{i\alpha L}$ we have $R(g) = e^{i\alpha R}$

$$e^{2i\xi A} \rightarrow e^{i\alpha L}\, e^{2i\xi A}\, e^{-i\alpha R}$$

or $\quad \Sigma \rightarrow \Omega_L \,\Sigma\, \Omega_R^{+}$

from which we built $\frac{f^2}{4} \mathrm{tr}\,(\partial_\mu \Sigma^+ \partial^\mu \Sigma)$

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Alternatively, let us construct p_μ

Exercise: extract $p_\mu A$ and $v_\mu V$ from

$e^{-i\xi A} \partial_\mu e^{i\xi A}$

and show that $\mathrm{tr}\,(p_\mu p^\mu) \approx \mathrm{tr}(\partial_\mu \Sigma^+ \partial^\mu \Sigma)$

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