S-matrix equivalence theorem

S-matrix elements do not depend on the choice of interpolating fields.
S-matrix elements are invariant under a local field redefinition:

set of all fields $\qquad \varphi \to \varphi + F(\varphi)$

$$F(\varphi) \equiv F(\varphi, \partial\varphi, \partial^2\varphi \dots \partial^N\varphi)$$

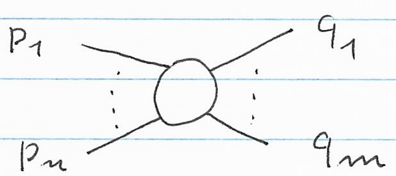
$\leftarrow$ finite

all evaluated at the same space-time point

sketch of the proof:

S-matrix elements are the residues of the Green's functions in momentum space.
Consider a theory depending on a single field φ describing a scalar particle of mass m.

To evaluate the S-matrix element associated to the scattering



we need the Green's function:

$$G(q_1 \dots q_m, p_1 \dots p_n) =$$

$$= \prod_{i=1}^{m} \int d^4y_i \, e^{i q_i y_i} \prod_{j=1}^{n} d^4x_j \, e^{-i p_j x_j} \times$$

$$\times \langle 0 | T(\varphi(y_1) \dots \varphi(y_m)\, \varphi(x_1) \dots \varphi(x_n)) | 0 \rangle$$

The function $G(q_1 \dots q_m p_1 \dots p_n)$ has $n+m$ poles at $q_i^2 = m^2$ $\quad p_j^2 = m^2$ $\quad q_i^0, p_j^0 > 0$

The S-matrix element, up to an overall factor, is given by

(LSZ or reduction formula)

$\langle q_1 \dots q_m | p_1 \dots p_n \rangle =$ Lehmann Symanzik Zimmermann (1955)

$$= \lim_{\substack{q_i^2 \to m^2 \\ q_i^0 > 0}} \lim_{\substack{p_j^2 \to m^2 \\ p_j^0 > 0}} \prod_{i=1}^{m} (q_i^2 - m^2) \prod_{j=1}^{n} (p_j^2 - m^2) \times$$
$$\times \; G(q_1 \dots q_m, p_1 \dots p_n)$$

Now consider the transformation:

$$\varphi \to \varphi + F(\varphi)$$

where we assume $F(\varphi)$ at least quadratic in φ (if linear $F(\varphi) = c_1 \varphi + d_1 \Box \varphi + \dots$ this corresponds to a wave function renormalization)

Specific example: $\quad F(\varphi) = c\,\varphi^2$

We show that this does not change the residues of G

Simplest Green's function is the propagator

$$\int d^4x\, e^{ipx} \langle 0|T(\varphi(x)\varphi(0))|0\rangle = \Delta(p)$$

$$\Delta(p) = \frac{iZ}{p^2 - m^2 + i\epsilon} + \dots$$

terms not contributing to the residue iZ

If we redefine the propagator

$$\Delta(p) \to \int d^4x\, e^{ipx}$$

$$\Big[\langle 0|T(\varphi(x)\varphi(0))|0\rangle$$
①

$$+ c\langle 0|T(\varphi^2(x)\varphi(0))|0\rangle + c\langle 0|T(\varphi(x)\varphi^2(0))|0\rangle$$
② ③

$$+ c^2\langle 0|T(\varphi^2(x)\varphi^2(0))|0\rangle\Big] \equiv \Delta'(p)$$
④

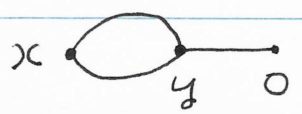
① Gives the original $\Delta(p)$

②, ③, ④ do not have a pole at $p^2 = m^2$ at least in perturbation theory

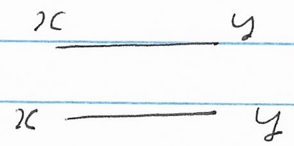
$\langle 0|T(\varphi^2(x)\varphi(0))|0\rangle =$ Wick contraction at tree level in the free theory $= 0$

$$+ \langle 0|T\varphi_0^2(x)\varphi_0(0)\,\lambda\varphi^3(y)|0\rangle$$

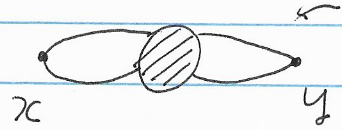
interaction



has no pole

$\langle 0|T(\varphi^2(x)\,\varphi^2(y))|0\rangle =$ 

disconnected diagrams

$+$ 

← no pole

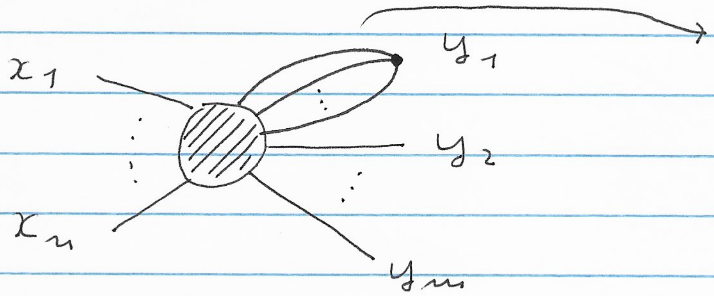
$\langle 0|T(\varphi(y_1)\,\dots\,\varphi(y_m)\,\varphi(x_1)\cdots\varphi(x_n))|0\rangle \to$

$\to$ same $+$

$+\ \langle 0|T\big(F(\varphi(y_1))\cdots\varphi(y_m)\,\varphi(x_1)\cdots\varphi(x_n)\big)|0\rangle + \dots$

does not have the same poles as original Green's function

It can be represented by a diagram:



→ N lines if $F(\varphi) \approx \varphi^N$

Fourier transform has lost a pole

Remark: Green's functions in general change

The S-matrix element are unchanged.

Field redefinition in EFT. Equivalent Lagrangians

$$\mathcal{L}_{IR} = \mathcal{L}_{LO} + \sum_i c_i^{(1)} \mathcal{L}_i^{(1)} + \sum_i c_i^{(2)} \mathcal{L}_i^{(2)} + \dots$$

such that $LO = O(0)$ $\quad c_i^{(p)} = O(p)$
in the power counting.
If the field redefinition is of the type:

$$\varphi \to \varphi + c_i^{(q)} F(\varphi) \qquad (1)$$

and our expansion ends at $p \geq q$
Then only $(-q+p+1)$ orders need to be modified:

$$LO, \; \dots, \; c_i^{(-q+p)}$$

Indeed, considering the shift (1) in the term of order $(-q+p+1)$ produces a term of order $(p+1) > p$.

In NRQED we stopped at order $(1/c^2)$ and the transformation

$$\varphi \to \varphi + \frac{1}{c^2} F(\varphi)$$

was only needed at LO.

Use of e.o.m. as field redefinitions

Assume we stop at order 1

$$\mathcal{L}_{IR} = \mathcal{L}_{LO} + \sum_i c_i^{(1)} \mathcal{L}_i^{(1)}$$

To be concrete, take $\mathcal{L}_{IR}(\varphi)$, φ real scalar field and:

$$\mathcal{L}_{LO} = \frac{1}{2} \partial\varphi \partial\varphi - \frac{1}{2} m^2 \varphi^2 - V_{LO}(\varphi)$$

The argument is essentially unchanged by considering more fields, spins, etc..

If $\qquad \mathcal{L}_j^{(1)} = F(\varphi)(\partial^2 + m^2)\varphi$

Then $\mathcal{L}_{IR}$ is equivalent (same S-matrix elements) to

$$\mathcal{L}'_{IR} = \mathcal{L}_{LO} + \sum_{i \neq j} c_i^{(1)} \mathcal{L}_i^{(1)} - c_j \frac{\delta V_{LO}}{\delta \varphi} F(\varphi)$$

Proof: take $\qquad \varphi \to \varphi + c_j F(\varphi)$

$$\delta S_{LO} = \delta \int d^4x \, \mathcal{L}_{LO}$$
$$= \int d^4x \, \frac{\delta \mathcal{L}_{LO}}{\delta \varphi} c_j F(\varphi)$$

$$\frac{\delta \mathcal{L}_{LO}}{\delta \varphi} = \text{Euler-Lagrange} = -(\partial^2 + m^2)\varphi - \frac{\partial V_{LO}}{\partial \varphi}$$

$$\delta S_{LO} = \int d^4x \left[-c_j (\partial^2 + m^2)\varphi \cdot F(\varphi) - c_j \frac{\partial V_{LO}}{\partial \varphi} \cdot F(\varphi) \right]$$

$\uparrow$

cancels $c_j \mathcal{L}_j^{(1)}$

→ we can use the LO eom to eliminate or transform an operator at 1st order.

→ In general: we can use the e.o.m. at order $p-q$ to eliminate/transform an operator at order q if the expansion stops at order p

Examples

NRQED

To eliminate $\dfrac{\xi}{mc^2} \bar{\chi} D_t^2 \chi$

we used $\chi \to \left[1 + \dfrac{\xi}{2mc^2}\left(iD_t - \dfrac{D^2}{2m}\right)\right]\chi$

to get the new operator $-\dfrac{\xi}{4m^3c^2} \bar{\chi} D^4 \chi$

Same result using the e.o.m.

$$\left(iD_t + \frac{D^2}{2m}\right)\chi = O\left(\frac{1}{c}\right)$$

$$\to \quad iD_t\,\chi = -\frac{D^2}{2m} + O\left(\frac{1}{c}\right)$$

$$\frac{\xi}{mc^2}\bar{\chi} D_t^2 \chi = -\frac{\xi}{mc^2}\bar{\chi}(-i\overleftarrow{D_t})(i\overrightarrow{D_t})\chi$$

$$= -\frac{\xi}{mc^2}\left[\bar{\chi}\frac{\overleftarrow{D}^2}{2m}\frac{D^2}{2m}\chi\right] + \ldots$$

$$= -\frac{\xi}{4m^3c^2}\bar{\chi} D^4 \chi + \ldots$$ much faster

Example:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4 + \frac{c_1}{\Lambda^2}\varphi^3\Box\varphi$$

$$+ \frac{c_6}{\Lambda^2}\varphi^6 + \dots$$

classical e.o.m : $\quad -(\Box + m^2)\varphi - \frac{\lambda}{6}\varphi^3 = O\left(\frac{1}{\Lambda^2}\right)$

$$\Box\varphi = -m^2\varphi - \frac{\lambda}{6}\varphi^3 + O\left(\frac{1}{\Lambda^2}\right)$$

$$\mathcal{L}' = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

$$- \frac{c_1}{\Lambda^2}\varphi^3\left(m^2\varphi + \frac{\lambda}{6}\varphi^3\right) + \frac{c_6}{\Lambda^2}\varphi^6 + \dots$$

$$= \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 - \left(\frac{\lambda}{4!} + c_1\frac{m^2}{\Lambda^2}\right)\varphi^4$$

$$+ \left(\frac{c_6}{\Lambda^2} - \frac{\lambda c_1}{6\Lambda^2}\right)\varphi^6 + \dots$$

Find the field redefinition giving rise to $\mathcal{L}'$. It is $\quad \varphi \to \varphi + \frac{c_1}{\Lambda^2}\varphi^3$

The independent parameters of $\mathcal{L}$ are 3: m^2 and the 2 combinations $\left(\frac{\lambda}{4} + c_1\frac{m^2}{\Lambda^2}\right) \equiv \lambda'$ and $\left(\frac{c_6}{\Lambda^2} - \frac{\lambda}{6}\frac{c_1}{\Lambda^2}\right) \equiv c_6'$

Using m^2, λ, c_1, c_6 provides a redundant parametrization, since physical quantities will depend just on λ', c_6' and m^2.

Exercise

$O_1 = \bar{\chi} \dfrac{D_k \vec{\sigma} \cdot \vec{B} D_k}{m^3} \chi$ $\qquad$ $O_2 = i \bar{\chi} \bar{\sigma} \cdot \dfrac{(\vec{B} \times \vec{B} - \vec{E} \times \vec{E})}{m^3} \chi$

check invariance under

- gauge
- P and T
- set power of C, normalization $\bar{\chi} i D_t \chi$.

	$\dfrac{iD_t}{c}$	iD_k	$\dfrac{E_k}{c}$	$\dfrac{B_k}{c}$	$\bar{\chi}\chi$	$\bar{\chi}\sigma^k\chi$
[ ]	ℓ^{-1}	ℓ^{-1}	ℓ^{-2}	ℓ^{-2}	.	.
P	+	−	−	+	+	+
T	$(-i)(-\frac{D_t}{c})$	$(-i)D_k$	+	−	+	−

c.c. since T is antilinear

	$\bar{\chi}$	D_k	$\bar{\sigma} \cdot \vec{B}$	$D_k \chi$	
gauge			o.k		
P	+	−	+	−	o.k.
T	−	+	−	+	o.k.

$i \bar{\chi} \bar{\sigma} \cdot (\vec{B} \times \vec{B} - \vec{E} \times \vec{E}) \chi$

gauge			o.k.		
P		+	+	+	o.k.
T	−	−	+	+	o.k.

$P X P^{-1} = -X$ $\qquad$ $T X T^{-1} = X$

$P p P^{-1} = -p$ $\qquad$ $T p T^{-1} = -p$

$[X, P] = i\hbar$ preserved $\qquad$ $[X, P] = +i\hbar$ preserved only if

$T i 1\!\!1 T^{-1} = -i 1\!\!1$

Set power of c:

2 ways

① power counting $\qquad \frac{iD_t}{c} \qquad \frac{E_k, B_k}{c} \qquad mc$

$$O_1 \to \frac{O_1}{c^4} \qquad\qquad O_2 \to \frac{O_2}{c^5}$$

Remember: to normalize $\bar{\chi} \frac{iD_t}{c} \chi \to \bar{\chi} i D_t \chi$

we need $\chi \to \sqrt{c}\,\chi$

Therefore:

$$\frac{O_1}{c^3} \qquad\qquad \frac{O_2}{c^4}$$

② dimension

$[mc] = \ell^{-1} \qquad [m] = \ell^{-2} t \qquad [eE] = [eB] = \ell^{-1} t^{-1}$

$[iD_t] = t^{-1}$

$$\left[\frac{D_k \sigma B D_k}{m^3 c^p}\right] = \ell^{-2} \ell^{-1} t^{-1} \ell^{6} t^{-3} \ell^{-p} t^{p} = t^{-1}$$
$$= \ell^{3-p} t^{p-4} = t^{-1} \qquad \to p = 3$$

$$\left[\frac{B \times B}{m^3 c^p}\right] = \ell^{-2} t^{-2} \ell^{6} t^{-3} \ell^{-p} t^{p} = t^{-1}$$
$$= \ell^{4-p} t^{p-5} = t^{-1} \qquad \to p = 4$$

Anti unitary operator T

$\langle x | T y \rangle \overset{d.}{\equiv} \langle U_T y | x \rangle$ where U_T is unitary

properties :

(1) $\langle Tx | Ty \rangle = \langle y | x \rangle = \overline{\langle x | y \rangle}$

$\langle Tx | Ty \rangle = \langle U_T y | U_T x \rangle = \langle y | x \rangle$ o.k.

(2) $T(\alpha |x\rangle + \beta |y\rangle) = \alpha^* T|x\rangle + \beta^* T|y\rangle$

$\langle z | T(\alpha |x\rangle + \beta |y\rangle) =$
$\langle U_T(\alpha x + \beta y) | z \rangle$
$= \alpha^* \langle U_T x | z \rangle + \beta^* \langle U_T y | z \rangle$
$= \alpha^* \langle z | T x \rangle + \beta^* \langle z | T y \rangle$ o.k.

$$\mathcal{L} = \varphi^* i\partial_t \varphi + \varphi^* \frac{\partial_x^2}{2m} \varphi$$ [redacted] $$c_2\, \varphi^*\varphi - c_4 (\varphi^*\varphi)^2$$

classify operators $c_{2,4}$ in d dimension

Scaling in d dimensions

$$t = e^{2\alpha} t' \qquad \varphi(x,t) = e^{-\Delta_\varphi \alpha} \varphi(x',t')$$
$$x = e^{\alpha} x'$$

$$\int dt\, d^{d-1}x \quad \varphi^* \left(i\partial_t + \frac{\partial_x^2}{2m}\right)\varphi$$

$$= \int dt' d^{d-1}x \quad \underbrace{\cancel{e^{2\alpha}}\, e^{(d-1)\alpha}\, e^{-2\Delta_\varphi \alpha}\, \cancel{e^{-2\alpha}}}_{e^{[d-1-2\Delta_\varphi]\alpha} \,=\, 1} \quad \varphi'^*(\;'\;)\varphi'$$

$$\Delta_\varphi \equiv \frac{d-1}{2}$$

$$\Delta_{\varphi^*\varphi} = (d-1)$$
$$\Delta_{(\varphi^*\varphi)^2} = 2d-2$$

relevant $\Delta < d$

marginal $\Delta = d$

irrelevant $\Delta > d$

$\varphi^*\varphi$ $\qquad d-1 < d$ $\qquad$ <u>sempre</u>

always relevant

$(\varphi^*\varphi)^2$	$2d-2 < d$	$d < 2$	relevant
	$2d-2 = d$	$d = 2$	marginal
	$2d-2 > d$	$d > 2$	irrelevant

Exercise

Consider the Lifshitz scaling:

$$t \to e^{z\alpha} t \qquad x \to e^{\alpha} x$$

in d-dimension

$z = 1$ Lorentz compatible

$z = 2$ Galilean group compatible

generators of time translation $P_0 = \partial_t$

spatial translation $P_i = \partial_i$

scaling transformation $D = -z\, t\partial_t - x^i \partial_i$

Subalgebra with P_0, P_i, D

$$[D, P_i] = P_i \qquad [D, P_0] = z\, P_0 \quad \leftarrow \text{derive this}$$

$$[(-z t \partial_t - x^i \partial_i)\partial_k + \partial_k (z t \partial_t + x^i \partial_i)]\, f$$

$= \dots\ \partial_k f$ etc.

Exercise

$$O_1 = \bar{\chi}\, \frac{D_k \vec{\sigma} \cdot \vec{B} D_k}{m^3}\, \chi \qquad O_2 = i \bar{\chi}\, \frac{\vec{\sigma} \cdot (\vec{B} \times \vec{B} - \vec{E} \times \vec{E})}{m^3}\, \chi$$

→ are they P, T invariant?

→ relative power counting $\frac{1}{c^{n_1}} \leftrightarrow \frac{1}{c^{n_2}}$

c^{n_1}/c^{n_2}

P o.k.

T $-\ +\ -\ +$ o.k.

Power counting $\frac{iD_t}{c}$, $\frac{e(E, B)}{c}$, mc

$$O_1 \cdot \frac{e}{c} \cdot \frac{1}{m^3 c^3} \qquad O_2 \cdot \frac{e^2}{c^2} \cdot \frac{1}{m^3 c^3}$$

$$c^{n_1}/c^{n_2} = c^4/c^5 = \frac{1}{c}$$

Notice that, by consistence, it is no more guaranteed that $(F_{ke})^2$ and $(F_{0e})^2$ are marginal operators

$$\int d^3x\, dt\, (F_{ke})^2 = \int d^3x'\, dt'\, e^{5\alpha} e^{-2\alpha} (\partial_{k'} e^{-\alpha} A_e')^2$$

$$= e^{\alpha} \int d^3x\, dt\, (F_{ke})^2$$

<u>relevant operator</u>

$$\int d^3x\, dt\, (F_{0e})^2 = \int d^3x'\, dt'\, e^{5\alpha} e^{-4\alpha} e^{-2\alpha} (F_{0e}')^2$$

<u>irrelevant op.</u> $\quad = e^{-\alpha} \int d^3x\, dt\, (F_{0e})^2$

Power counting for non-relativistic theories
Consider for instance:

$$\mathcal{L} = i\varphi^* \partial_t \varphi + \varphi^* \frac{\nabla^2}{2m}\varphi - c_2 \varphi^*\varphi - c_4(\varphi^*\varphi)^2$$

$$\to \quad i\partial_t \varphi = -\frac{\nabla^2}{2m}\varphi + c_2\varphi + \dots$$ Schroedinger-like equation

Determine dimensions for fields and operators.
Time and space cannot scale in the same way. Consider $d=4$ and

$$\begin{cases} t = e^{2\alpha} t' \\ \vec{x} = e^{\alpha} \vec{x}' \end{cases}$$

Now $d^4x = e^{5\alpha} d^4x'$ and the canonical dimension of φ is $3/2$.
We have

$5 > [\mathcal{O}]$ relevant
$5 = [\mathcal{O}]$ marginal
$5 < [\mathcal{O}]$ irrelevant

$[\varphi^*\varphi] = 3$ relevant
$[(\varphi^*\varphi)^2] = 6$ irrelevant (This was marginal in relativistic QFT)

Relation to a relativistic real scalar φ_R is

$$\varphi_R = \frac{1}{2m}\left[e^{-imt}\varphi + e^{+imt}\varphi^*\right]$$

can be found in Kaplan [K]

Example III Why the sky is blue?

Basic reason: blue light scatters more strongly than red light with the atoms in the atmosphere.

Consider the Rayleigh scattering: low-energy scattering of photons with neutral atoms in their ground state.

Low energy condition:

$$E_\gamma \ll \Delta E \approx m_e \alpha^2 \approx 10\,eV$$

red $\approx 1.8\,eV$ satisfied by

blue $\approx 3.1\,eV$ visible light

Approximation: we treat the scattering as elastic and we neglect atom recoils (infinitely heavy atoms) $\equiv$ NR limit for atoms

Effective theory: most general Lagrangian describing elastic photon-atom scattering compatible with Lorentz and gauge invariance.

Photons are described by the em field A_μ

gauge transformation $A_\mu' = A_\mu + \partial_\mu \alpha$

Field strenght $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

gauge invariant

Atoms are described by a (complex) field

φ_v destroyes an atom with 4-velocity

φ_v^+ creates v^μ $v^\mu v_\mu = 1$

Effective theory for Rayleigh scattering

① d.o.f. neutral atom (1 specie) φ (complex)

photons γ $A_\mu = (\varphi, A_k)$

② symmetries

→ Translations, 3D rotations

→ gauge invariance

→ conservation of number N of atoms

Lagrangian depending on $\varphi^* \varphi$

$\varphi \to e^{i\beta} \varphi$

→ P, T invariance

③ Power counting : in a moment

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \varphi^* i \partial_t \varphi + \varphi^* \frac{\partial_k^2}{2m} \varphi$$

$$+ c_1 \varphi^* \varphi E_k E_k + c_2 \varphi^* \varphi B_k B_k + \dots$$

We cannot write $\varphi^* i D_t \varphi$ or $\varphi^* \frac{D_k^2}{2m} \varphi$ since atoms are neutral

dimensions $[\partial_t] = [\partial_k^2/2m] = t^{-1}$

$[E_k] = [B_k] = \ell^{-1} t^{-1}$

→ $[c_{1,2}](\ell^{-1} t^{-1})^2 = t^{-1}$

→ $[c_{1,2}] = \ell^2 t^1$

assume $c_{1,2}$ is a combination of a mass M and a velocity c

---

Remark : no minimal coupling $\varphi^* \frac{D_k^2}{2m} \varphi$ since φ is e.m. neutral and $D_k \equiv \partial_k$.

$$c_i = \frac{1}{M^p c^q}$$

$$[M] = \ell^{-2} t \qquad [c] = \ell\, t^{-1}$$

$$[M^p c^p] = \ell^{-2p+q}\, t^{p-q} = \ell^{-2}\, t^{-1}$$

$$\begin{aligned} p - q &= -1 \\ -2p + q &= -2 \\ \hline -p &= -3 \end{aligned} \quad \rightarrow \quad p = 3 \qquad q = 4$$

$$\boxed{c_i \approx \frac{1}{M^3 c^4}}$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \varphi^* i \partial_t \varphi + \varphi^* \frac{\partial_k^2}{2m} \varphi$$

$$+ \frac{\tilde{c}_1}{M^3 c^4} \varphi^* \varphi\, E_k E_k + \frac{\hat{c}_2}{M^3 c^4} \varphi^* \varphi\, B_k B_k + \dots$$

Which scale is M?
We have many scales:

$$E_\gamma \ll \Delta E = m_e \alpha^2 \ll a_0^{-1} = m_e \alpha \ll m_A$$

If γ have no access to the atomic structure the cross section is nearly classical and will mainly depend on the size a_0 of the target:

$$\boxed{M \approx a_0^{-1} = m_e \alpha}$$

On dimensional grounds:

$$\sigma(\gamma A \to \gamma A) \propto \frac{E_\gamma^n}{M^6 c^8} \approx \frac{E_\gamma^4}{M^6 c^{10}} \approx \frac{a_0^6 \omega^4}{c^{10}}$$

imposing $[\sigma] = \ell^{+2}$