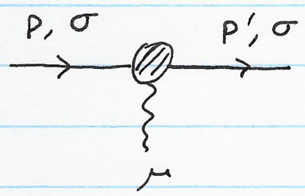
Possible application of NRQED

Proton radius puzzle

Electromagnetic interaction with a on-shell proton described by:



$$\equiv -ie\, \bar{u}_{\sigma'}(p')\, \Gamma^\mu(p,p')\, u_\sigma(p)$$

$$\Gamma^\mu(p,p') = \gamma^\mu F_1(q^2) + i \frac{\sigma^{\mu\nu}}{2m_p}(p'-p)_\nu F_2(q^2)$$

normalization $F_1(0) = 1$

$F_1(q^2)$, $F_2(q^2)$ are real, adimensional

In perturbation theory: $\begin{cases} F_1(q^2) = 1 + \dots \\ F_2(q^2) = \dots \end{cases}$ $\mathcal{O}(\alpha)$

We can combine $F_{1,2}(q^2)$ into an electric form factor G_E and a magnetic one G_M.

$$G_E = F_1 + q^2 \frac{F_2}{4m_p^2} \qquad G_M = F_1 + F_2$$

The proton radius r_E^p is defined as the slope of $G_E(q^2)$ at $q^2 = 0$:

$$\boxed{\begin{aligned} (r_E^p)^2 &\equiv 6 \left. \frac{dG_E(q^2)}{dq^2} \right|_{q^2=0} \\ &= 6 \left[F_1'(0) + \frac{F_2(0)}{4m_p^2} \right] \end{aligned}}$$

r_E^p can be measured in 4 ways:

① ordinary hydrogen ($e^- p$) spectroscopy

② muonic hydrogen ($\mu^- p$) spectroscopy

③ $e^- p$ scattering → less precise

④ $\mu^- p$ scattering → MUSE experiment at PSI (2019)

	①	②
r_E^p	0.8751(61) fm	0.84184(67) fm

5σ

proton radius puzzle

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In ② r_p is extracted from $\Delta E(2S-2P)$

Recall: $\Delta E(2S-2p) = 0$ with fine structure only in ordinary hydrogen 1st measured in 1947 by Lamb (Lamb shift)

$$\overset{①}{\Delta} E(2S-2P) \approx 4\,\mu eV$$
$$\approx h\nu \qquad \nu = 1058\ MHz$$

in ② $\Delta E(2S-2p)$ receives many contributions

$$\Delta E(2S-2P) = \frac{2\pi}{3}\alpha\,|\varphi_{200}(0)|^2 (r_E^p)^2 c + \dots$$

$$|\varphi_{200}(0)|^2 = \frac{1}{n^3 \pi a_0^3}\Bigg|_{n=2} = \frac{m_r^3 \alpha^3 c^3}{8\pi}$$

$E_n \equiv -\dfrac{E_I}{n^2}$ $\qquad E_I \equiv \dfrac{1}{2} m_p c^2 \alpha^2$ $\qquad a_0 \equiv \dfrac{1}{m_e c \alpha}$

$$\Delta E(2S-2P) = \frac{1}{12}\,\frac{\alpha c}{a_0}\left(\frac{r_p}{a_0}\right)^2 + \cdots$$

$$= \frac{1}{12}\, m_e c^2 \alpha^2 \left(\frac{r_p}{a_0}\right)^2 + \cdots$$

$$= \frac{E_I}{6}\left(\frac{r_p}{a_0}\right)^2 + \cdots$$

$a_0 \sim 0.5\ 10^{-10}\,m \qquad r_p \sim 10^{-15}\,m \sim 1\,fm$

$\left(\frac{r_p}{a_0}\right)^2 \sim 4 \times 10^{-10}$

in ordinary hydrogen: $\frac{E_I}{6}\left(\frac{r_p}{a_0}\right)^2 \simeq 10^{-3}\,\mu eV$

$\ll \Delta E(2S-2P) \simeq 4\,\mu eV$

in muonic hydrogen: $a_0 \sim \frac{1}{m_{lepton}}$

$$\left(\frac{r_p}{a_0}\right)^2_\mu \Big/ \left(\frac{r_p}{a_0}\right)^2_e \approx \left(\frac{m_\mu}{m_e}\right)^2$$

$$\frac{(E_I)_\mu}{(E_I)_e} \simeq \left(\frac{m_\mu}{m_e}\right)$$

$\rightarrow$ r_p contribution expected to be $\left(\frac{m_\mu}{m_e}\right)^3 \approx 10^7$ bigger

$$10^{-3}\,\mu eV \times 10^7 \simeq 10^4\,\mu eV \sim 10\,meV$$

The theoretical prediction being:

$$\Delta E(2S-2P) \cong 206\,meV - 5\,r_p^2\,\frac{meV}{fm^2} + \cdots$$

Why measuring r_E^p from a muonic atom is advantageous?

$$a_0 \propto \frac{1}{m_e}$$

$$\rightarrow \quad \Delta E(r)_\mu = \underbrace{\left(\frac{m_\mu}{m_e}\right)^3}_{\text{almost } 10^7 \text{ enhancement}} \Delta E(r)_e$$

From the theory view point the best way to compute energy levels is using NRQED
Remember:

$$\mathcal{L} = \chi^+ \Big[\Big(i D_t + c_2 \frac{D^2}{2m} \Big) \chi + c_F \frac{e}{2mc} \sigma \cdot B$$
$$+ c_D \frac{e}{8m^2c^2} (\mathbf{D} \cdot E - E \cdot \mathbf{D})$$
$$+ i c_S \frac{e}{8m^2c^2} \sigma \cdot (D \times E - E \times D) + \dots \Big] \chi$$

here χ is the proton

relation between c coefficients and form factors:

$$c_F = F_1(0) + F_2(0) = G_M(0) \qquad c_S = F_1(0) + 2F_2(0)$$

$$c_D = F_1(0) + 2F_2(0) + 8M^2 F_1'(0)$$
$$= F_1(0) + 8m^2 \underbrace{\left[F_1'(0) + \frac{F_2(0)}{4m^2} \right]}_{\frac{(r_E^p)^2}{6}}$$
$$= F_1(0) + \frac{4}{3} m_p^2 (r_E^p)^2$$

$$\boxed{c_D = 1 + \frac{4}{3} m_p^2 (r_E^p)^2}$$

$c_D - 1$ is a measure of $(r^p)^2$ in units of m_p^{-2}

Effective field theory for the Rayleigh scattering
(or "why the sky is blue")?

Basic reason: blue light scatters more strongly than red light with the atoms in the atmosphere.

Formally:
low-energy scattering of photons with neutral atoms in their ground state
(Rayleigh scattering)

low-energy condition:

$$E_\gamma \ll E_I \approx m_e \alpha^2 \approx 10\,eV$$

satisfied by visible light,

red $\approx$ 1.8 eV   blue $\simeq$ 3.1 eV

Approximations
→ Elastic scattering $\gamma A \to \gamma A$
→ neglect Atom recoil : $m_A \to \infty$
(justifies a NR limit for atoms)

Effective theory

① d.o.f.   neutral atom A $\leftrightarrow$ φ (complex field)
photon $\gamma \leftrightarrow A_\mu = (\varphi, A_i)$

② Symmetries
- Translations, rotations in 3D
- gauge invariance
- conservation of number N of atoms
global $\varphi \to e^{i\alpha}\varphi$
→ Lagrangian depends on $\varphi^* \varphi$
- P, T invariance

③ Power counting: should be guessed

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \varphi^* i \partial_t \varphi + \varphi^* \frac{\partial_k^2}{2m} \varphi$$

ordinary derivatives since φ is neutral

$$+ C_1 \varphi^* \varphi E_k E_k + C_2 \varphi^* \varphi B_k B_k + \dots$$

We can determine the dimensions of $C_{1,2}$

$$[\partial_t] = \left[\frac{\partial_k^2}{2m}\right] = t^{-1} \qquad [E_k] = [B_k] = \ell^{-1} t^{-1}$$

$$\rightarrow \quad [C_{1,2}]\, \ell^{-2} t^{-2} = t^{-1} \qquad \rightarrow \quad [C_{1,2}] = \ell^2 t$$

Assume C_i are combination of a mass M and a velocity c (of light)

$$C_i \approx \frac{1}{M^p c^q} \qquad [Mc] = \ell^{-1} \qquad [c] = \ell\, t^{-1}$$

$$\ell^2 t = [C_i] = \ell^p \, \ell^{p-q} \, t^{q-p}$$
$$= \ell^{2p-q} \, t^{q-p}$$

$$q - p = 1$$
$$2p - q = 2 \qquad \rightarrow \quad p = 3 \qquad q = 4 \qquad C_i \approx \frac{1}{M^3 c^4}$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \varphi^* i \partial_t \varphi + \varphi^* \frac{\partial_k^2}{2m} \varphi$$

$$+ \frac{\tilde{C}_1}{M^3 c^4} \varphi^* \varphi E_k E_k + \frac{\tilde{C}_2}{M^3 c^4} \varphi^* \varphi B_k B_k$$

Which scale is M ?

We have many scales

$$E_\gamma \ll E_I \approx m_e \alpha^2 \ll a_0^{-1} m_e \alpha \ll m_A$$

If γ has no access to the atomic structure the cross-section is nearly classical and will mainly depend on the size a_0 of the target

$$M \approx a_0^{-1} \approx m_e \alpha$$

$$A(\gamma A \to \gamma A) \approx \frac{\tilde{c}_i}{M^3 c^4}$$

dimension

$l^2 t$

$$\sigma(\gamma A \to \gamma A) \propto \frac{|\tilde{c}_i|^2}{M^6 c^8} \cdot \xi$$

$$l^4 t^2 [\xi] = l^2$$

$$\to [\xi] = l^{-2} t^{-2}$$

ξ can only be a combination of E_γ and c

$$[E_\gamma^p c^q] = t^{-p} l^q t^{-q} = l^{-2} t^{-2}$$

$$= t^{-p-q} l^q = l^{-2} t^{-2}$$

$\to q = -2 \qquad -p + 2 = -2 \qquad \to p = +4$

$$\xi \approx \frac{E_\gamma^4}{c^2}$$

$$\to \boxed{\sigma(\gamma A \to \gamma A) \approx |\tilde{c}|^2 \cdot \frac{E_\gamma^4}{M^6 c^{10}}}$$

Indeed the larger E_γ, the bigger $\sigma(\gamma A \to \gamma A)$.

where does our approximation breaks down?

Power counting involves: $\frac{1}{M} \simeq \frac{1}{(m_e \alpha)} \simeq \frac{1}{(1\,keV)}$

break down at $E_\gamma \approx m_e \alpha \approx 1\,keV$?

No, much before! When $E_\gamma \sim m_e \alpha^2 \approx 10\,eV$ light will excite the atom to an higher level and the scattering is nomore elastic.

power counting	cut-off
$\frac{1}{(m_e \alpha)}$	$m_e \alpha^2$