Loops in EFT

So far we have ignored them. They can modify previous discussion in two ways.

1. In top-down approach they can give rise to additional effects (new operators) not arising in the tree-level approximation.
2. They can affect the scaling of operators, that in general will deviate from the scaling analyzed in the tree-level approximation.

Deviations from the naive scaling typically give rise to $\log \frac{M}{\mu}$ $\begin{array}{l} \leftarrow \text{heavy} \\ \leftarrow \text{light} \end{array}$ corrections to the coefficients of the operators in the EFT.

If $M \gg \mu$ these corrections can be large, even in a perturbative (i.e. small coupling) regime. They can be efficiently accounted for by renormalization group techniques.

We have seen that under rescaling

$$x^\mu = s\, x'^\mu \qquad (\text{previously } s \equiv e^\alpha)$$

$$\frac{a}{\Lambda^{-\Delta}}\, O \longrightarrow \frac{a}{\Lambda^{-\Delta}}\, s^\Delta O$$

where Δ in the tree-level approximation is entirely determine by the dimension of O: $\Delta \equiv d - [O]$

Taking into account loops, typically

$$\boxed{a \text{ becomes } \quad a\left(1 + \frac{b\overset{\text{some coupling}}{\alpha}}{4\pi} \log s\right)}$$

This can be interpreted as a correction to the naive scaling:

$$a\left(1+\frac{b\alpha}{4\pi}\log s\right)=a\left(1+\log s^{\frac{b\alpha}{4\pi}}\right)$$

$$\approx a\, s^{\frac{b\alpha}{4\pi}}$$

so that:

$$\frac{a}{\Lambda^{-\Delta}}\,\mathcal{O} \xrightarrow{x^\mu = s x'^\mu} \frac{a}{\Lambda^{-\Delta}}\, s^{\left(\Delta+\frac{b\alpha}{4\pi}\right)}\,\mathcal{O}$$

Depending on the sign of b a marginal operator $\mathcal{O}$ can become:

relevant $b > 0$

irrelevant $b < 0$

in our classification.

Moreover it is important to accurately account for these corrections when moving from high to low energies (large s limit).

We will first discuss point 1 through a concrete example.

Low energy EFT for QED ($E \ll m_e$)

Here the heavy particle is the electron.

$$\mathcal{L}_{UV} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m_e)\Psi$$

$$D_\mu \Psi = (\partial_\mu + ieA_\mu)\Psi$$

$$e^{iS_{IR}(A)} = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi}\, e^{iS_{UV}(A,\Psi,\bar{\Psi})}$$

Tree-level:
the eom for the electron field are

$$(i\gamma^\mu D_\mu - m_e)\Psi = 0$$

and, when we substitute this back into S_{UV}, we get:

$$\mathcal{L}_{IR}(A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \mathcal{L}_0(A)$$

the free theory for the electromagnetic field.
To find something non trivial we should go beyond the tree-level approximation.
We could directly evaluate:

$$\int \mathcal{D}\Psi \mathcal{D}\bar{\Psi}\, e^{iS_{UV}(A,\Psi,\bar{\Psi})} \propto \det(i\not{D} - m_e) \times e^{iS_0(A)}$$

since this is a Gaussian integral.
This leads to a non-local Lagrangian

for the e.m. field A_μ, that we can expand in inverse powers of the heavy scale m_e:

$$\mathcal{L}_{IR}(A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \sum_{i,n=1}^{\infty} \frac{C_i^{(4+n)}}{m_e^n} O_i^{(4+n)}$$

Strategy: instead of evaluating

$$e^{iS_{IR}(A)} \propto e^{iS_0(A)} \det(i\not{D} - m_e)$$

we will identify the first few terms in the expansion exploiting Lorentz invariance and gauge invariance, C and P. Then we will compute the coefficients $C_i^{(4+n)}$ by a matching procedure, i.e. asking that IR and UV theories agree on their predictions.

Lorentz invariance: saturate Lorentz indices in covariant formalism

gauge invariance: $A_\mu \to A_\mu + \partial_\mu \alpha$

gauge invariance combination: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

P : $A_\mu(x) \to A^\mu(x_P)$ metric $(+\,-\,-\,-)$

C : $A_\mu(x) \to -A_\mu(x)$

P : $F_{\mu\nu} \to F^{\mu\nu}$ C : $F_{\mu\nu} \to -F_{\mu\nu}$

P accounted for by Lorentz. We will enforce CP.

$n=1$ (dim 5 operators) : none

No way of saturate all Lorentz indices

applies to all odd n

$n = 2$ (dim 6 operators)

$F_{\mu\rho} F^{\rho\sigma} F_\sigma{}^\mu$ not invariant under CP

$\left\{\begin{array}{l} \partial^\mu F_{\mu\nu}\, \partial^\rho F_\rho{}^\nu \\ F_{\mu\nu}\, \Box F^{\mu\nu} \\ \vdots \end{array}\right.$ not needed. Why?
When we put on-shell one of these e.m. fields their contribution to the amplitude vanishes

on-shell $\equiv$ solution of e.o.m.

$$\partial_\mu F^{\mu\nu} = \emptyset$$

$\rightarrow$ $\Box A_\mu = 0$ plus $\partial_\mu A^\mu = 0$

We will come back to this point.

$n = 4$ (dim 8 operators)

We have two non-trivial independent operators:

$(F_{\mu\nu} F^{\mu\nu})^2$ and $(F_{\mu\rho} F^{\rho\nu})(F^{\mu\sigma} F_{\sigma\nu})$

For convenience we stop at this order and we parametrize $\mathcal{L}_{IR}(A)$ as

$$\mathcal{L}_{IR}(A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + a_1 \frac{\alpha^2}{16 m_e^4} (F_{\mu\nu} F^{\mu\nu})^2$$

$$+ a_2 \frac{\alpha^2}{16 m_e^4} (F_{\mu\rho} F^{\rho\nu})(F^{\mu\sigma} F_{\sigma\nu}) + \ldots$$

higher orders

$\alpha \equiv \frac{e^2}{4\pi}$

α^2 expected by loop counting

Even before computing $a_{1,2}$ we have already non-trivial consequences.

→ Due to the new operators Maxwell equations are modified and field equations are no more linear.

→ The photon is now self-interacting

We can estimate the photon-photon cross section

$A(\gamma\gamma \to \gamma\gamma)$ is dimensionless (exercise)

$\approx \frac{\alpha^2}{16 m_e^4} E_\gamma^4$ meaningful range $E_\gamma < m_e$

$$\boxed{\sigma \approx \left(\frac{\alpha^2}{16 m_e^4}\right)^2 E_\gamma^6 \quad E_\gamma < m_e}$$

since $[\sigma] = -2$

put $E_\gamma \approx m_e$ $\quad \sigma \sim \left(\frac{\alpha^2}{16}\right)^2 \cdot \frac{1}{m_e^2} \approx 4.2 \; 10^{-11} \; MeV^{-2}$

$(\hbar c)^2 \cong$ ~~xxxxxxx~~ $0.4 \; GeV^2 \; m\,barn$

$= 0.4 \, (10^3)^2 \; MeV^2 \; m\,barn$

$\sigma \approx 1.7 \; 10^{-5} \; m\,barn \approx 17 \, nb \left(\frac{E_\gamma}{m_e}\right)^6 = 10^{-32} \, cm^2 \left(\frac{E_\gamma}{m_e}\right)^6$

$$\boxed{\text{if we take } E_\gamma \approx m_e/10 \quad \sigma \approx 17 \, fb}$$

γ-rays $\approx 1 \, MeV$ $\quad 10^{-30} \, cm^2$

visible light $\approx 1 \, eV$ $\quad 10^{-66} \, cm^2$

$1 \, barn = 10^{-24} \, cm^2$

Such a small cross section has not been detected yet.

Exercise

Show that

$$\frac{\alpha^2}{16}\left[a_1 (F_{\mu\nu}F^{\mu\nu})^2 + a_2 (F_{\mu\rho}F^{\rho\nu}F^{\mu\sigma}F_{\sigma\nu})\right]$$

can also be written as

$$\frac{\alpha^2}{16}\left[a_1' (F_{\mu\nu}F^{\mu\nu})^2 + a_2' (F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right]$$

$$\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$$

$\varepsilon^{\mu\nu\rho\sigma}$ Levi-Civita tensor

$\varepsilon^{0123} = -\varepsilon_{0123} = +1$

Notice that $F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ & 0 & -B^3 & B^2 \\ & & 0 & -B^1 \\ & & & 0 \end{pmatrix}$

$\tilde{F}^{\mu\nu}$ is obtained from $F^{\mu\nu}$ by the substitution $(E \to B,\ B \to -E)$

$(F^{\mu\nu}F_{\mu\nu}) = -2(\vec{E}^2 - \vec{B}^2)$ CP-even

$(F_{\mu\nu}\tilde{F}^{\mu\nu}) = -4(\vec{E}\cdot\vec{B})$ CP-odd

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

$$(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 = \frac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta}F_{\mu\nu}F_{\rho\sigma}F_{\alpha\beta}F_{\gamma\delta}$$

Identity: $\varepsilon^{\mu\nu\rho\sigma}\varepsilon^{\alpha\beta\gamma\delta} = -\det(g^{\lambda\lambda'})$

$\lambda = \mu\nu\rho\sigma$
$\lambda' = \alpha\beta\gamma\delta$

$= 24$ terms

$= -(g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}g^{\sigma\delta} + \ldots$ minus sign for odd number of substitutions $)$

Possible structures

$$-\left(g^{\mu\alpha}g^{\nu\beta}g^{\rho\gamma}g^{\sigma\delta} + g^{\mu\alpha}g^{\nu\gamma}g^{\rho\delta}g^{\sigma\beta} + g^{\mu\alpha}g^{\nu\delta}g^{\rho\beta}g^{\sigma\gamma}\right)\times$$

$\times\, 8$ (accounting for $(\alpha\leftrightarrow\beta)$, $(\gamma\leftrightarrow\delta)$, $(\alpha\beta\leftrightarrow\gamma\delta)$ exchanges)

$$(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 = \frac{1}{4}(-8)\Big[(F_{\mu\nu}F^{\mu\nu})^2 - 2\,(F_{\mu\rho}F^{\rho\nu}F^{\mu\sigma}F_{\sigma\nu})\Big]$$

$$\rightarrow \frac{\alpha^2}{16}\left(a_1'\,(F_{\mu\nu}F^{\mu\nu})^2 + a_2'\,(F_{\mu\nu}\tilde{F}^{\mu\nu})^2\right)$$

$$= \frac{\alpha^2}{16}\Big(a_1'(F_{\mu\nu}F^{\mu\nu})^2 - 2a_2'\,(F_{\mu\nu}F^{\mu\nu})^2 + 4a_2'(F_{\mu\rho}F^{\rho\nu}F^{\mu\sigma}F_{\sigma\nu})\Big)$$

$$= \frac{\alpha^2}{16}\left(a_1 (F_{\mu\nu}F^{\mu\nu})^2 + a_2 (F_{\mu\rho}F^{\rho\nu}F^{\mu\sigma}F_{\sigma\nu})\right)$$

$$\rightarrow \boxed{\begin{aligned} a_1 &= a_1' - 2a_2' \\ a_2 &= 4a_2' \end{aligned}}$$

or $a_2' = \frac{a_2}{4}$ $\qquad a_1' = a_1 + \frac{2}{4} a_2$

cfr Falkowski

$$\boxed{a_1 = -\frac{4}{9} \qquad a_2 = +\frac{56}{45}}$$

$$\rightarrow a_1' = \frac{8}{45} \qquad a_2' = \frac{14}{45}$$

$$\boxed{\begin{aligned} \frac{a_1'}{16} &= \frac{8}{720} = \frac{1}{90} \\ \frac{a_2'}{16} &= \frac{14}{720} = \frac{7}{360} \end{aligned}}$$

Take home message: the basis of invariants (in this case under Lorentz + gauge + CP) at a given dimension is not uniquely fixed. This should be taken into account when comparing different $\mathcal{L}_{IR}$.