Euler - Heisenberg EFT

SUMMARY

$$\mathcal{L}_{UV} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi$$

$\mathcal{L}_{IR}$ in BU approach built from

① A_μ
② Poincaré, $U(1)_{em}$ gauge, P, C
③ $1/m$

$$\mathcal{L}_{IR}(A) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a_1 \alpha^2}{16 m^4} (F_{\mu\nu} F^{\mu\nu})^2 + \frac{a_2 \alpha^2}{16 m^4} (F_{\mu\rho} F^{\rho\nu} F_{\nu\sigma} F^{\sigma\mu}) + \ldots$$

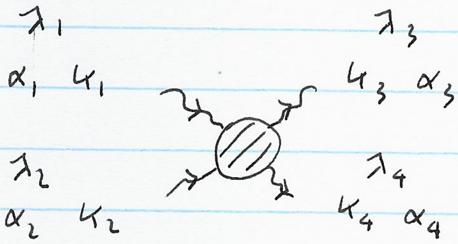
$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a_1' \alpha^2}{16 m^4} (F_{\mu\nu} F^{\mu\nu})^2 + \frac{a_2' \alpha^2}{16 m^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 + \ldots \qquad \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

$$a_1 = a_1' - 2a_2'$$
$$a_2 = 4a_2'$$

Today: $a_1 = -\frac{4}{9}$ $\qquad a_2 = +\frac{56}{45}$

$a_1' = \frac{8}{45}$ $\qquad a_2' = \frac{14}{45}$

To evaluate a_1, a_2 we match two amplitudes $A(\gamma\gamma \to \gamma\gamma)$



here momenta k_1, k_2 are incoming, $k_{3,4}$ outgoing ($k_3 \to -k_3$ $\quad k_4 \to -k_4$ for the unphysical amplitude)

To simplify the computation we choose a specific kinematics:

$$k_3 = +k_1 \qquad\qquad k_4 = +k_2 \qquad\qquad \leftrightarrow \quad t = 0$$

In terms of Mandelstam variables:

$s = (k_1 + k_2)^2 = 2k_1k_2$

$t = (k_1 - k_3)^2 = 0$

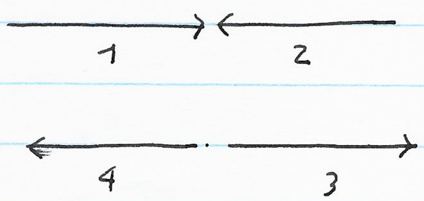
$u = (k_1 - k_4)^2 = (k_1 - k_2)^2 = -2k_1k_2 \qquad\qquad s + t + u = 0$ o.k.

$t = 0$ <u>forward scattering</u>

In CM frame:

$k_1 = (k, 0, 0, k) \qquad\qquad k_2 = (k, 0, 0, -k)$

$k_3 = (-k, 0, 0, -k) \qquad\qquad k_4 = (-k, 0, 0, +k)$



Moreover we can focus on the part of $A(\gamma\gamma \to \gamma\gamma) \propto g_{..}g_{..} \leftrightarrow$ Lorentz indices carried by metric tensor

In the CM frame the polarization vectors read

$$\epsilon^{\alpha_1}_{\lambda_1}(k_1) = (0, -\lambda_1, -i, 0)/\sqrt{2} \qquad \lambda_i = \pm 1$$

$$\epsilon^{\alpha_2}_{\lambda_2}(k_2) = (0, +\lambda_2, -i, 0)/\sqrt{2}$$

$$\epsilon^{\alpha_3 *}_{\lambda_3}(k_3) = (0, -\lambda_3, i, 0)/\sqrt{2}$$

$$\epsilon^{\alpha_4 *}_{\lambda_4}(k_4) = (0, +\lambda_4, i, 0)/\sqrt{2}$$

$$\rightarrow \quad \epsilon_\lambda(k)\cdot k' \equiv 0 \quad \text{whatever } k, k'$$

$$A_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} = \epsilon^{*\alpha_4}_{\lambda_4}\, \epsilon^{*\alpha_3}_{\lambda_3}\, \epsilon^{\alpha_2}_{\lambda_2}\, \epsilon^{\alpha_1}_{\lambda_1}\, M_{\alpha_1\alpha_2\alpha_3\alpha_4}(k_1,k_2,k_3,k_4)$$

$$\rightarrow \quad M_{\alpha_1\alpha_2\alpha_3\alpha_4} = A(s)\, g_{\alpha_1\alpha_3}\, g_{\alpha_2\alpha_4} + B(s)\, g_{\alpha_1\alpha_4}\, g_{\alpha_2\alpha_3}$$

$$+ C(s)\, g_{\alpha_1\alpha_2}\, g_{\alpha_3\alpha_4} + \ldots$$

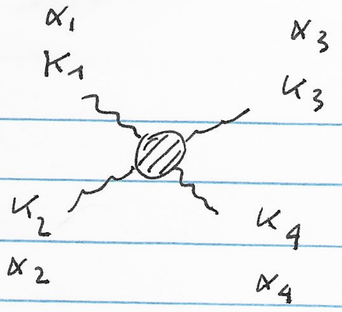
↳ terms not contributing to A

In the IR side we get: $\mathcal{M}_{\alpha_1\alpha_2\alpha_3\alpha_4} =$

$$i \frac{\alpha^2}{2 m_e^4} \Big[(4 a_1 + a_2) (g_{\alpha_1\alpha_2} g_{\alpha_3\alpha_4} + g_{\alpha_1\alpha_4} g_{\alpha_2\alpha_3}) + 2 a_2 \, g_{\alpha_1\alpha_3} g_{\alpha_2\alpha_4} \Big] (k_1 \cdot k_2)^2 + \dots$$

only invariant

UV side



$$= \frac{\alpha^2}{\cancel{16} m_e^4} a_1 \times (\cancel{36}\,\overset{2}{2}\, i) \times$$

$$\Big[(K_1 \cdot K_2)(K_3 \cdot K_4)\, g_{\alpha_1\alpha_2}\, g_{\alpha_3\alpha_4} +$$
$$(K_1 \cdot K_3)(K_2 \cdot K_4)\, g_{\alpha_1\alpha_3}\, g_{\alpha_2\alpha_4} +$$
$$(K_1 \cdot K_4)(K_2 \cdot K_3)\, g_{\alpha_1\alpha_4}\, g_{\alpha_2\alpha_3} -$$
$$- (K_1 \cdot K_2)\, K_{3\alpha_4} K_{4\alpha_3}\, g_{\alpha_1\alpha_2} - (K_3 \cdot K_4)\, K_{1\alpha_2} K_{2\alpha_1}\, g_{\alpha_3\alpha_4}$$
$$- (K_1 \cdot K_3)\, K_{2\alpha_4} K_{4\alpha_2}\, g_{\alpha_1\alpha_3} - (K_2 \cdot K_4)\, K_{1\alpha_3} K_{3\alpha_1}\, g_{\alpha_2\alpha_4}$$
$$- (K_1 \cdot K_4)\, K_{2\alpha_3} K_{3\alpha_2}\, g_{\alpha_1\alpha_4} - (K_2 \cdot K_3)\, K_{1\alpha_4} K_{4\alpha_1}\, g_{\alpha_2\alpha_3}$$
$$+ K_{1\alpha_2} K_{2\alpha_1} K_{3\alpha_4} K_{4\alpha_3} + K_{1\alpha_3} K_{3\alpha_1} K_{2\alpha_4} K_{4\alpha_2}$$
$$+ K_{1\alpha_4} K_{4\alpha_1} K_{2\alpha_3} K_{3\alpha_2} \Big]$$

$$+ \frac{\alpha^2}{16 m_e^4} a_2 \times (4 i) \times$$

$$\Big[2 (K_1 \cdot K_2)(K_3 \cdot K_4)\left(g_{\alpha_1\alpha_3} g_{\alpha_2\alpha_4} + g_{\alpha_2\alpha_3} g_{\alpha_1\alpha_4} \right)$$
$$+ 2 (K_1 \cdot K_3)(K_2 \cdot K_4)\left(g_{\alpha_1\alpha_2} g_{\alpha_3\alpha_4} + g_{\alpha_1\alpha_4} g_{\alpha_2\alpha_3} \right)$$
$$+ 2 (K_1 \cdot K_4)(K_2 \cdot K_3)\left(g_{\alpha_1\alpha_2} g_{\alpha_3\alpha_4} + g_{\alpha_1\alpha_3} g_{\alpha_2\alpha_4} \right)$$
$$+ 2 K_{1\alpha_2} K_{2\alpha_3} K_{3\alpha_4} K_{4\alpha_1} + 2 K_{1\alpha_2} K_{2\alpha_4} K_{3\alpha_1} K_{4\alpha_3}$$
$$+ 2 K_{1\alpha_3} K_{2\alpha_1} K_{3\alpha_4} K_{4\alpha_2} + 2 K_{1\alpha_3} K_{2\alpha_4} K_{3\alpha_2} K_{4\alpha_1}$$
$$+ 2 K_{1\alpha_4} K_{2\alpha_1} K_{3\alpha_2} K_{4\alpha_3} + 2 K_{1\alpha_4} K_{2\alpha_3} K_{3\alpha_1} K_{4\alpha_2}$$
$$+ 2 (K_1 \cdot K_2)\big(K_{3\alpha_1} K_{4\alpha_2} g_{\alpha_3\alpha_4} + K_{3\alpha_2} K_{4\alpha_1} g_{\alpha_3\alpha_4}$$
$$- K_{3\alpha_4} K_{4\alpha_2} g_{\alpha_1\alpha_3} - K_{3\alpha_4} K_{4\alpha_1} g_{\alpha_2\alpha_3}$$
$$- K_{3\alpha_2} K_{4\alpha_3} g_{\alpha_1\alpha_4} - K_{3\alpha_1} K_{4\alpha_3} g_{\alpha_2\alpha_4} \big)$$
$$+ 2 (K_1 \cdot K_3)\big(K_{2\alpha_1} K_{4\alpha_3} g_{\alpha_2\alpha_4} + K_{2\alpha_3} K_{4\alpha_1} g_{\alpha_2\alpha_4}$$
$$- K_{2\alpha_4} K_{4\alpha_3} g_{\alpha_1\alpha_2} - K_{2\alpha_3} K_{4\alpha_2} g_{\alpha_1\alpha_4}$$
$$- K_{2\alpha_4} K_{4\alpha_1} g_{\alpha_2\alpha_3} - K_{2\alpha_1} K_{4\alpha_2} g_{\alpha_3\alpha_4} \big)$$
$$+ 2 (K_1 \cdot K_4)\big(K_{2\alpha_1} K_{3\alpha_4} \cancel{g_{\alpha_2\alpha_3}} + K_{2\alpha_4} K_{3\alpha_1} g_{\alpha_2\alpha_3}$$
$$- K_{2\alpha_3} K_{3\alpha_4} g_{\alpha_1\alpha_2} - K_{2\alpha_4} K_{3\alpha_2} g_{\alpha_1\alpha_3}$$
$$- K_{2\alpha_3} K_{3\alpha_1} g_{\alpha_2\alpha_4} - K_{2\alpha_1} K_{3\alpha_2} g_{\alpha_3\alpha_4} \big)$$

$$
\begin{aligned}
&+2(k_2\cdot k_3)\big(k_{1\alpha_2}\, k_{4\alpha_3}\, g_{\alpha_1\alpha_4} + k_{1\alpha_3} k_{4\alpha_2} g_{\alpha_1\alpha_4} \\
&\qquad - k_{1\alpha_4} k_{4\alpha_3}\, g_{\alpha_1\alpha_2} - k_{1\alpha_4} k_{4\alpha_2} g_{\alpha_1\alpha_3} \\
&\qquad - k_{1\alpha_3} k_{4\alpha_1}\, g_{\alpha_2\alpha_4} - k_{1\alpha_2} k_{4\alpha_1}\, g_{\alpha_3\alpha_4}\big) \\
&+2(k_2\cdot k_4)\big(k_{1\alpha_2}\, k_{3\alpha_4}\, g_{\alpha_1\alpha_3} + k_{1\alpha_4} k_{3\alpha_2} g_{\alpha_1\alpha_3} \\
&\qquad - k_{1\alpha_3} k_{3\alpha_4}\, g_{\alpha_1\alpha_2} - k_{1\alpha_3} k_{3\alpha_2} g_{\alpha_1\alpha_4} \\
&\qquad - k_{1\alpha_4} k_{3\alpha_1}\, g_{\alpha_2\alpha_3} - k_{1\alpha_2}\, k_{3\alpha_1}\, g_{\alpha_3\alpha_4}\big) \\
&+2(k_3\cdot k_4)\big(k_{1\alpha_3}\, k_{2\alpha_4}\, g_{\alpha_1\alpha_2} + k_{1\alpha_4} k_{2\alpha_3} g_{\alpha_1\alpha_2} \\
&\qquad - k_{1\alpha_2} k_{2\alpha_4} g_{\alpha_1\alpha_3} - k_{1\alpha_2} k_{2\alpha_3} g_{\alpha_1\alpha_4} \\
&\qquad - k_{1\alpha_4} k_{2\alpha_1} g_{\alpha_2\alpha_3} - k_{1\alpha_3} k_{2\alpha_1}\, g_{\alpha_2\alpha_4}\big)\Big]
\end{aligned}
$$

symmetries

$i \leftrightarrow j \;\equiv\; k_i \leftrightarrow k_j$ and $\alpha_i \leftrightarrow \alpha_j$

Ward identity:

$k_i^{\alpha_i}(\ldots\ldots) = 0$

checked for 1st term

2nd term

$k_1^{\alpha_1}$:

$$
\begin{aligned}
&(1\cdot2)(3\cdot4)(1_3\, g_{24} + g_{23}\, 1_4) \\
&+ (1\cdot3)(2\cdot4)(1_2\, g_{34} + 1_4\, g_{23}) \\
&+ (1\cdot4)(2\cdot3)(1_2\, g_{34} + 1_3\, g_{24}) \\
&+ \cancel{(1\cdot4)\, 1_2\, 2_3\, 3_4} + \cancel{(1\cdot3)\, 1_2\, 2_4\, 4_3} \\
&+ (1\cdot2)\, \cancel{1_3\, 3_4\, 4_2} + \cancel{(1\cdot4)\, 1_3\, 2_4\, 3_2} \\
&+ (1\cdot2)\, \cancel{1_4\, 3_2\, 4_3} + \cancel{(1\cdot3)\, 1_4\, 2_3\, 4_2} \\
&+ (1\cdot2)\big((1\cdot3)\, 4_2\, g_{34} + (1\cdot4)\, 3_2\, g_{34} - \cancel{3_4\, 4_2\, 1_3} \\
&\qquad - 3_4\, (1\cdot4)\, g_{23} - \cancel{3_2\, 4_3\, 1_4} - (1\cdot3)\, 4_3\, g_{24}\big) \\
&+ (1\cdot3)\big((1\cdot2)\, 4_3\, g_{24} + (1\cdot4)\, 2_3\, g_{24} - \cancel{2_4\, 4_3\, 1_2} \\
&\qquad - \cancel{2_3\, 4_2\, 1_4} - (1\cdot4)\, 2_4\, g_{23} - (1\cdot2)\, 4_2\, g_{34}\big) \\
&+ (1\cdot4)\big((1\cdot2)\, 3_4\, g_{23} + (1\cdot3)\, 2_4\, g_{23} - \cancel{2_3\, 3_4\, 1_2} \\
&\qquad - \cancel{2_4\, 3_2\, 1_3} - (1\cdot3)\, 2_3\, g_{24} - (1\cdot2)\, 3_2\, g_{34}\big)
\end{aligned}
$$

$= 0$ o.k.

$\boxed{\text{This is the result in the most general kinematics}}$

$$
\begin{aligned}
4a_1 [& (1\cdot 2)(3\cdot 4) \quad g_{12}\, g_{34} \\
+ & (1\cdot 3)(2\cdot 4) \quad g_{13}\, g_{24} \\
+ & (1\cdot 4)(2\cdot 3) \quad g_{14}\, g_{23}] \\
+ a_2 [& (1\cdot 2)(3\cdot 4) \quad (g_{13}\, g_{24} + g_{14}\, g_{23}) \\
+ & (1\cdot 3)(2\cdot 4) \quad (g_{12}\, g_{34} + g_{14}\, g_{23}) \\
+ & (1\cdot 4)(2\cdot 3) \quad (g_{12}\, g_{34} + g_{13}\, g_{24})]
\end{aligned}
$$

Enforce momentum conservation and eliminate 4: $1+2+3+4=0 \quad \to \quad 4 = -1-2-3$

$$
\begin{aligned}
(1\cdot 2)(3\cdot 4) &= -(1\cdot 2)(1\cdot 3) - (1\cdot 2)(2\cdot 3) - \cancel{(1\cdot 2)(3\cdot 3)} \\
(1\cdot 3)(2\cdot 4) &= -(1\cdot 3)(1\cdot 2) - \cancel{(1\cdot 3)(2\cdot 2)} - (1\cdot 3)(2\cdot 3) \\
(1\cdot 4)(2\cdot 3) &= -\cancel{(1\cdot 1)(2\cdot 3)} - (1\cdot 2)(2\cdot 3) - (1\cdot 3)(2\cdot 3)
\end{aligned}
$$

Independent terms are $(1\cdot 2)(1\cdot 3)$ $\quad (1\cdot 2)(2\cdot 3)$ $\quad (1\cdot 3)(2\cdot 3)$

becomes:

$$
\begin{aligned}
& (1\cdot 2)(1\cdot 3) [(-4a_1 - a_2)\, g_{12} g_{34} \\
& \qquad (-4a_1 - a_2)\, g_{13} g_{24} \\
& \qquad - 2a_2\, g_{14}\, g_{23}] \\
+ & (1\cdot 2)(2\cdot 3) [(-4a_1 - a_2)\, g_{12}\, g_{34} \\
& \qquad (-4a_1 - a_2)\, g_{14}\, g_{23} \\
& \qquad -2a_2\, g_{13}\, g_{24}] \\
+ & (1\cdot 3)(2\cdot 3) [(-4a_1 - a_2)\, g_{13} g_{24} \\
& \qquad (-4a_1 - a_2)\, g_{14}\, g_{23} \\
& \qquad - 2a_2\, g_{12}\, g_{34}]
\end{aligned}
$$

since $4^2 = 0 \qquad (1+2+3)^2 = 0$

$$(1\cdot 2) + (2\cdot 3) + (1\cdot 3) = 0$$

and we can also eliminate e.g. $(2\cdot 3)$

We get:

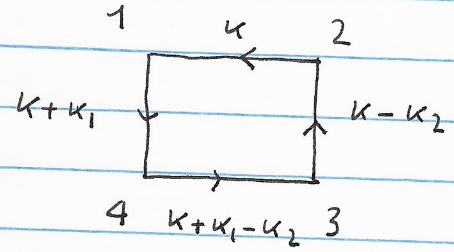
$$
\begin{aligned}
& g_{12} g_{34} [(4a_1 + a_2)(1\cdot 2)^2 + 2a_2 (1\cdot 3)^2 + 2a_2 (1\cdot 2)(1\cdot 3)] \\
+ & g_{13}\, g_{24} [(4a_1 + a_2)(1\cdot 3)^2 + 2a_2 (1\cdot 2)^2 + 2a_2 (1\cdot 2)(1\cdot 3)] \\
+ & g_{14}\, g_{23} [(4a_1 + a_2)(1\cdot 2)^2 + (4a_1 + a_2)(1\cdot 3)^2 + 8a_1 (1\cdot 2)(1\cdot 3)]
\end{aligned}
$$

We have 6 topologies

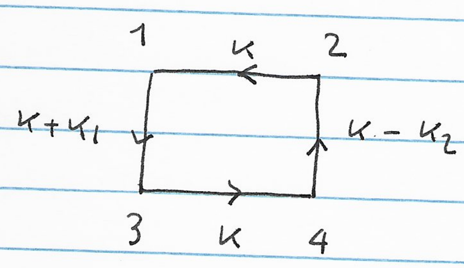
$k_3 = -k_1$ $k_4 = -k_2$

kinematics

(1)



(2)



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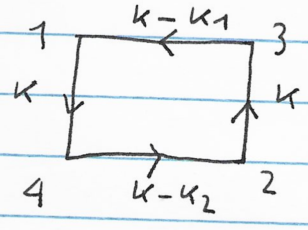

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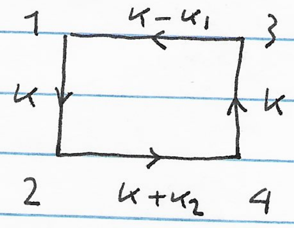
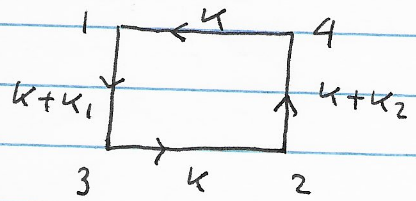
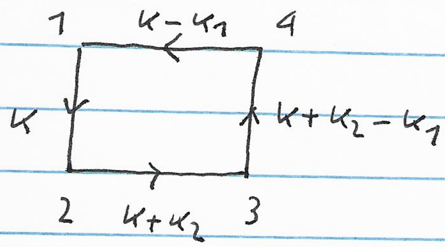


(5)



(6)

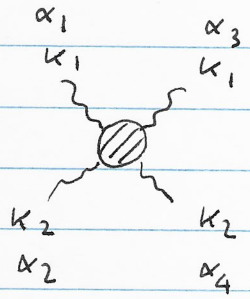


Diagrams (1,6) (3,5) (2,4) give the same result: they only differ by trace routing and $(k_1, k_2) \to (-k_1, -k_2)$

choose a specific kinematics to match:

$-k_1 = k_3 \qquad -k_2 = k_4$

choose the $(k\cdot k)(k\cdot k)$ for matching



$$= i\frac{\alpha^2}{2m_e^4}\Big[4a_1\,(k_1\cdot k_2)^2\,(g_{\alpha_1\alpha_2}g_{\alpha_3\alpha_4} + g_{\alpha_1\alpha_4}g_{\alpha_2\alpha_3})$$
$$+ a_2\,(k_1\cdot k_2)^2\,(g_{\alpha_1\alpha_3}g_{\alpha_2\alpha_4} + g_{\alpha_2\alpha_3}g_{\alpha_1\alpha_4} + g_{\alpha_1\alpha_2}g_{\alpha_3\alpha_4} + g_{\alpha_1\alpha_3}g_{\alpha_2\alpha_4})$$
$$+ \dots\Big]$$

$$= i\frac{\alpha^2}{2m_e^4}(k_1\cdot k_2)^2\Big[(4a_1+a_2)(g_{\alpha_1\alpha_2}g_{\alpha_3\alpha_4} + g_{\alpha_1\alpha_4}g_{\alpha_2\alpha_3})$$
$$+ 2a_2(g_{\alpha_1\alpha_3}g_{\alpha_2\alpha_4} + \cancel{g_{\alpha_1\alpha_3}g_{\alpha_2\alpha_4}})\Big] + \dots$$

$\rightarrow$ Compute the 1-loop amplitude in QED and extract

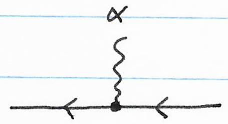
- the term $\dfrac{(k_1\cdot k_2)^2}{m_e^4}$ in the low-energy expansion valid for $|k_1\cdot k_2| \ll m_e^2$
- the coefficient of $g_{\alpha_1\alpha_2}g_{\alpha_3\alpha_4} \propto 4a_1 + a_2$
  the coefficient of $g_{\alpha_1\alpha_3}g_{\alpha_2\alpha_4} \propto 2a_2$

$k_1 + k_2 + k_3 + k_4 = 0 \qquad k_3 = -k_1$

$\cancel{2k_1}\cdot \qquad\qquad k_4 = -k_2$

Diagram (1) Feynman rules

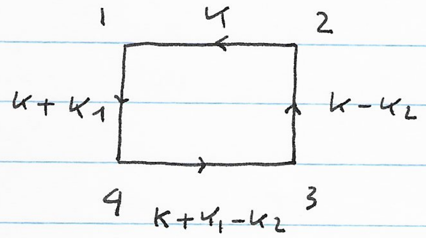
$$= \frac{i}{\not{k} - m}$$



$$= (-ie)\,\gamma_\alpha$$

(-1) for fermion loop

$(-ie)^4 \equiv e^4 \qquad (-i)^4 \equiv 1$



$$= -e^4 \int \frac{d^d k}{(2\pi)^d} \, \mathrm{tr}\Big[\gamma_{\alpha_1} \frac{1}{\not{k}-m} \gamma_{\alpha_2} \frac{1}{\not{k}-\not{k}_2-m} \gamma_{\alpha_3} \cdot \frac{1}{\not{k}+\not{k}_1-\not{k}_2-m} \gamma_{\alpha_4} \frac{1}{\not{k}+\not{k}_1-m}\Big]$$

$$= -e^4 \int \frac{d^d k}{(2\pi)^d} \frac{\mathrm{tr}\left[\gamma_{\alpha_1}(\not{k}+m)\gamma_{\alpha_2}(\not{k}-\not{k}_2+m)\gamma_{\alpha_3}(\not{k}+\not{k}_1-\not{k}_2+m)\gamma_{\alpha_4}(\not{k}+\not{k}_1+m)\right]}{(k^2-m^2)((k-k_2)^2-m^2)((k+k_1-k_2)^2-m^2)((k+k_1)^2-m^2)}$$

$$= -e^4 I_1$$

The integral and its low momentum expansion can be evaluated with Mathematica using Package X software.

$$I_1 = \frac{i}{16\pi^2} \frac{1}{m^4} \left[\frac{62}{315} g_{\alpha_1\alpha_2} g_{\alpha_3\alpha_4} - \frac{26}{105} g_{\alpha_1\alpha_3} g_{\alpha_2\alpha_4}\right] (k_1 \cdot k_2)^2$$

Similarly for the other diagrams.
In units of $i \frac{e^4}{16\pi^2} \frac{1}{m^4} (k_1 \cdot k_2)^2 \equiv i \frac{\alpha^2}{m^2} (k_1 \cdot k_2)^2$

DIAGRAMS	$g_{\alpha_1\alpha_2} g_{\alpha_3\alpha_4}$	$g_{\alpha_1\alpha_3} g_{\alpha_2\alpha_4}$
1, 6	$-\frac{62}{315}$	$\frac{26}{105}$
2, 4	$+\frac{1}{7}$	$\frac{59}{315}$
3, 5	$-\frac{5}{63}$	$\frac{59}{315}$
TOTAL(×2)	$-\frac{4}{15}$	$+\frac{56}{45}$

Matching condition:

$$i \frac{\alpha^2}{m^2} (k_1 \cdot k_2)^2 \left[-\frac{4}{15} (g_{\alpha_1\alpha_2} g_{\alpha_3\alpha_4} + \dots) + \frac{56}{45} g_{\alpha_1\alpha_3} g_{\alpha_2\alpha_4} \right] + \dots$$

$$= i \frac{\alpha^2}{m^2} (k_1 \cdot k_2)^2 \left[\left(2a_1 + \frac{a_2}{2}\right)(g_{\alpha_1\alpha_2} g_{\alpha_3\alpha_4} + \dots) + a_2 \, g_{\alpha_1\alpha_3} g_{\alpha_2\alpha_4} \right]$$

$$\rightarrow \boxed{a_1 = -\frac{4}{9} \qquad a_2 = +\frac{56}{45}}$$

$$C_1 = \alpha^2 \left(a_1 + \frac{a_2}{2}\right) = \alpha^2 \left(-\frac{4}{9} + \frac{28}{45}\right) = \frac{8}{45} \alpha^2$$

$$C_2 = \alpha^2 \frac{a_2}{4} = \frac{14}{45} \alpha^2$$

① Suppose in the loop we have a <u>fermion</u> with mass M, electric charge Q (in units of e) and N_c colors.

We get the same result with the replacement

$$\frac{\alpha^2}{m^4} \to N_c \frac{\alpha^2 Q^4}{M^4}$$

② If in the loop we have a scalar with mass M_s, electric charge Q_s and N_c colors

$$\frac{\alpha^2}{m^4} \to N_c \frac{\alpha^2 Q_s^4}{M_s^4} \quad \text{and} \quad \boxed{a_1^s = +\frac{1}{18} \qquad a_2^s = +\frac{2}{45}}$$

<u>check</u> define $a^2 - b^2 = E^2 - B^2 = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu}$

$$ab = E \cdot B = -\frac{1}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

$$= \frac{e^4}{360\pi^2 m^4} \left[\frac{1}{4} (F^2)^2 + 7 \frac{1}{16} (F\tilde{F})^2 \right] \quad \text{spinor QED}$$

or $\frac{\alpha^2}{16 m^4} \left[\frac{8}{45} (F^2)^2 + \frac{14}{45} (F\tilde{F})^2 \right]$

$$\stackrel{!}{=} \frac{\alpha^2 \, \cancel{16\pi^2}^{\,\cancel{8}\,2}}{\cancel{360}_{\cancel{90}\,45} \cancel{\pi^2} m^4} \left[\cdots \right] = \frac{\alpha^2}{m^4 16} \cdot \frac{32}{45} \left[\cdots \right]$$

$$= \frac{\alpha^2}{16 m^4} \left[\frac{\cancel{32}\,8}{45 \cdot \cancel{4}} (F^2)^2 + \frac{\cancel{32}^{\,2}}{45} \frac{7}{\cancel{16}} (F\tilde{F})^2 \right]$$

O.K.

scalar QED

$$\frac{\cancel{e^4}\,\alpha\, 16\cancel{\pi^2}}{5760 m^4 \cancel{\pi^2}}\left[\frac{7}{4}(F^2)^2 + \frac{4}{16}(F\tilde{F})^2\right]$$

$$= \frac{\alpha}{360 m^4}\left[\frac{7}{4}(F^2)^2 + \frac{1}{4}(F\tilde{F})^2\right]$$

$$= \frac{\alpha^2}{16 m^4}\left[\underbrace{\frac{2}{45}\frac{7}{4}}_{\frac{14}{180}\ a_1'}(F^2)^2 + \frac{2}{45}\frac{1}{4}(F\tilde{F})^2\right]$$

$$\frac{2}{180} \quad a_2'$$

$$a_1 = \frac{14-4}{180} = +\frac{1}{18} \qquad a_2 = \frac{\cancel{4}\cdot 2}{\cancel{180}_{45}} = \frac{2}{45} \qquad \text{o.k.}$$

③ $\mathcal{L}_{UV} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial_\mu h)^2 - \frac{m_h^2}{2}h^2 + b_h \frac{\alpha}{4\pi f} hF\tilde{F}$

$\mathcal{L}_{IR}$ (integrating out h at the tree-level assuming $E \ll m_h$)

$$-m_h^2 h + b_h \frac{\alpha}{4\pi f}F\tilde{F} = 0$$

$$h = \frac{b_h}{m_h^2}\cdot\frac{\alpha}{4\pi f}F\tilde{F}$$

$$\boxed{a_1 = -\frac{b_h^2}{\pi^2} \qquad a_2 = +2\frac{b_h^2}{\pi^2}}$$

back into $\mathcal{L}_{UV}$:

$$\left[-\frac{\cancel{m_h^2}}{2}\frac{b_h^2}{m_h^{\cancel{4}}}\frac{\alpha^2}{(4\pi f)^2} + \frac{b_h}{m_h^2}\frac{\alpha^2}{(4\pi f)^2}\right]F\tilde{F}$$

$$= \frac{1}{2}\frac{b_h^2\alpha^2}{m_h^2\, 16\pi^2 f^2}F\tilde{F} \rightarrow \frac{1}{m^4} \rightarrow \frac{1}{m_h^2 f^2}$$

$$a_2' = \frac{b_h^2}{2\pi^2}$$