Advanced Topics in the Theory of Fundamental Interactions

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1. An $O(4)$ invariant theory describes the interactions of 4 real scalar fields h, π^a $(a = 1, 2, 3)$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu h \partial^\mu h + \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a - V(h, \pi)$$

$$V(h, \pi) = -\frac{\mu^2}{2}(h^2 + \vec{\pi}^2) + \frac{\lambda}{4}(h^2 + \vec{\pi}^2)^2 \qquad\qquad \lambda, \mu^2 > 0$$

The scalar potential V has a minimum at $h = v = \sqrt{\mu^2/\lambda}$ and $\vec{\pi} = 0$. Define $\sigma = h - v$ and derive the mass spectrum of the theory

2. By neglecting terms higher than quadratic in the field σ, derive the low-energy effective theory of the fields $\vec{\pi}$, at the three level and for $m_\sigma \gg E \gg m_{\vec{\pi}}$. In the relevant expansion keep terms containing up to 4 π^a and up to 2 derivatives, included (that is $(\vec{\pi}^2)^2$ and $\vec{\pi}^2 \Box \vec{\pi}^2$).

3. The low-energy Lagrangian can be used to evaluate, at the TL, the amplitude for the $\pi\pi$ scattering. Such an amplitude has the general form:

$$\mathcal{A}(\pi^a \pi^b \to \pi^c \pi^d) = A(s, t, u)\delta_{ab}\delta_{cd} + B(s, t, u)\delta_{ac}\delta_{bd} + C(s, t, u)\delta_{ad}\delta_{bc}$$

where $s = (k_a + k_b)^2$, $t = (k_a - k_c)^2$, $u = (k_a - k_d)^2$. Making use of the invariance under the two crossing the symmetries

$$\begin{cases} a \leftrightarrow c & k_a \leftrightarrow -k_c & \qquad (I) \\ a \leftrightarrow d & k_a \leftrightarrow -k_d & \qquad (II) \end{cases}$$

show that the amplitude depends on a single function of the Mandelstam variables s, t and u, for instance $A(s, t, u)$, enjoying a discrete symmetry.

4. Expand $A(s,t,u)$ in powers of s, t and u, keeping only terms up to the first order:

$$A(s,t,u) = A_0 + A_1^s s + A_1^t t + A_1^u u + ...$$

Using the previous results, including points 1. and 2., show that $A(s,t,u)$ depends on a single combination of the four constants A_0, $A_1^{s,t,u}$.

Determine the parametric dependence of such a constant on λ and v.

Determine the parametric dependence of the cross-section $\sigma(\pi^a\pi^b \to \pi^c\pi^d)$ on λ, v and the c.o.m. energy E.

5. The initial Lagrangian is deformed into

$$\begin{aligned} V(h,\pi) &= -\frac{\mu^2}{2}(h^2+\vec{\pi}^2) + \frac{\lambda}{4}(h^4 + (\pi^1)^4 + (\pi^2)^4 + (\pi^3)^4) \\ &+ \frac{2\rho}{4}(h^2\vec{\pi}^2 + (\pi^1)^2(\pi^2)^2 + (\pi^1)^2(\pi^3)^2 + (\pi^2)^2(\pi^3)^2) \\ &\qquad\qquad \lambda, \rho, \mu^2 > 0 \end{aligned}$$

such that $\rho = \lambda$ gives back the original theory. The one-loop beta functions for the quartic interaction of a scalar theory

$$-\frac{\lambda_{abcd}}{24}\varphi_a\varphi_b\varphi_c\varphi_d$$

are

$$16\pi^2\beta_{abcd} = \frac{1}{8}\sum_{e,f}(\lambda_{abef}\lambda_{efcd} + \lambda_{acef}\lambda_{efbd} + \lambda_{adef}\lambda_{efbc})$$

Identify λ_{aaaa} and λ_{aabb} $(a \neq b)$ in terms of λ and ρ and derive the one-loop RGE for λ and ρ.

6. What is the symmetry of the new theory?

Derive the 1-loop RGE for the ratio $r = \rho/\lambda$ and discuss the asymptotic IR behavior for UV initial conditions $r_{UV} > 1$ and $r_{UV} < 1$.