Non-relativistic QED

$$\mathcal{L}_{QED} = -\frac{1}{2} F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\not{D} - m)\Psi$$

a triumph of QFT

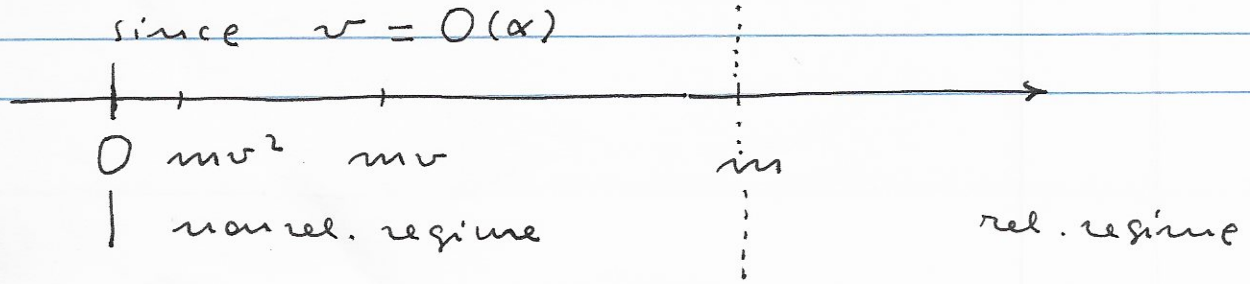
1947: anomalous magnetic moment of e^-
Lamb shift $= \Delta E(2S_{1/2} - 2P_{1/2})$ in hidrogen atom

$n = 2$ $\qquad E_n = -\frac{1}{4}\frac{\alpha^2}{n^2}$ $\qquad n = 1, 2, \dots$

$\deg = n^2 = 4$ $\qquad \begin{cases} n = 2 & l = 0 & 1 \text{ state} \\ n = 2 & l = 1 & 3 \text{ state} \end{cases}$

degenerate even in the Dirac-theory

However full relativistic QED not completely suitable for a number of application in atomic systems. We need NRQED.
Advantage: incorporate the small radiative corrections of relativistic QED into the non relativistic framework of boundstates like hydrogen atoms or positronium (e^+e^-)
Problem with QED applied to e.g. hydrogen
α enters in two ways:
1. Expansion parameter for QED loop corrections
2. Definition of the relevant scales
m_e = electron mass
$m_e v$ = hydrogen momentum
$m_e v^2$ = " energy
since $v = O(\alpha)$



More efficient strategy:
Keep α and v as independent parameters.
Define an EFT for $v \ll c$ where relativistic effects have been "integrated out"

$$\mathcal{L}_{EFT} = \sum_n C_n(\alpha) \left(\frac{v}{c}\right)^n O^{(n)}$$

describe the non-rel limit good to treat bound states

accounts for rad. corr. to the desired order

Main application: precise determination of energy levels of bound states like atoms, positronium, etc..
Extension to strong interaction is conceptually very similar NRQCD: description of bound states like $\Upsilon\ (b\bar{b})$ or $J/\psi\ (c\bar{c})$.
$v^2 \simeq 0.1$ $\qquad$ $v^2 \simeq 0.3$

Also: incorporation of rel. effects in many electrons atoms and in solids.

<u>These lectures:</u>

Work out $\mathcal{L}_{EFT}$ to order $\left(\frac{v}{c}\right)^2$

$n = 2$ suitable to incorporate e.g. fine structure effects in hydrogen atoms

with $C_n(\alpha) = C_n^{(0)} + C_n^{(1)}\alpha + C_n^{(2)}\alpha^2 + \dots$
at LO i.e. only $C_n^{(0)}$.

2 lectures:

- general strategy
- details

$\boxed{\text{Start with an exercise}}$

reintroduce factors of c in Dirac Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi$$

$$D_\mu \Psi = (\partial_\mu + ieA_\mu)\Psi \qquad \text{here } e < 0$$

I use Dirac representation of γ matrices

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \epsilon^{0123} = +1$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \qquad \sigma^{0i} = i\gamma^0\gamma^i = i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$\sigma^{ij} = \epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

Now $[l] \neq [t]$, we still set $\hbar = 1$

$\rightarrow \quad [p] = l^{-1} \qquad [E] = t^{-1}$

we use l and t as units

Lorentz force in Lorentz-Heaviside units

$$F = e\left(E + \frac{v}{c} \times B\right)$$

$\rightarrow \quad [eE] = [eB] = l^{-1} t^{-1}$

$E = \nabla\varphi \qquad B = \nabla \times A$

$\rightarrow \quad [e\varphi] = [eA] = t^{-1}$

Covariant derivative

$$[D_t \equiv (\partial_t + ie\varphi)] = t^{-1}$$
$$[D_k \equiv \partial_k + i\frac{e}{c}A_k] = l^{-1}$$

$$[D_\mu] = l^{-1} \qquad D_\mu \equiv \left(\frac{D_t}{c}, D_k\right)$$

$$D_\mu = \left(\frac{\partial_t}{c}, \partial_k\right) + i\frac{e}{c}(\varphi, A_k)$$

Kinetic term:

$$\bar\Psi\, i \not{D}\, \Psi = \bar\Psi \underbrace{\left(i\gamma^0 \frac{D_t}{c} + i\gamma^k D_k\right)}_{l^{-1}} \Psi$$

Mass term

$$-\bar\Psi \underbrace{mc}_{l^{-1}} \Psi$$

$$\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar\Psi\left(i\gamma^0 \frac{D_t}{c} + i\gamma^k D_k - mc\right)\Psi}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \text{originally}$$
$$[F_{kl} = \partial_k A_l - \partial_l A_k] = l^{-1}[A_k]$$
$$\left[F_{0l} = \frac{\partial_0}{c} A_l - \partial_l A_0\right] = l^{-1}[A_l] = l^{-1}[A_0]$$

$$\rightarrow \boxed{\begin{aligned} F_{0l} &= \frac{\partial_t}{c} A_l - \partial_l A_0 \\ F_{kl} &= \partial_k A_l - \partial_l A_k \end{aligned}}$$

Action $S = \int dt\, d^3x\, \mathcal{L}$ is dimensionless

$\to [\mathcal{L}] = t^{-1} \ell^{-3}$

since $\mathcal{L} = \bar{\Psi} (\ell^{-1}) \Psi$

so far $[\bar{\Psi}\Psi] = t^{-1}\ell^{-2}$

at the end we will redefine

$$\Psi \to \sqrt{c}\,\Psi$$

so that $[c\bar{\Psi}\Psi] = t^{-1}\ell^{-2} = (t^{-1}\ell)\,\ell^{-3}$

and Ψ has the simple unit $\ell^{-3/2}$.

Non-relativistic QED

① d.o.f.

We do not need e^- and e^+ as in the relativistic theory. We focus on e^- (e.g. hydrogen atom) described by a 2-component spinor χ:

$$\Psi \equiv \begin{bmatrix} \chi \\ \hline \eta \end{bmatrix}$$

NRQED includes e^- and the photon γ described by χ and φ, A_k.

~~② Symmetries~~

~~gauge invariance~~

~~gauge invariant / covariant combinations~~

~~$F_{\mu\nu}$, $\frac{D_t}{c} = \frac{\partial_t}{c} + \frac{ie}{c}\varphi$, $D_k = \partial_k + \frac{ie}{c}A_k$~~

~~Parity $\vec{x} \to -\vec{x}$~~

	~~D_t~~	~~D_k~~	~~$F_{k\ell}$~~	~~$F_{0\ell}$~~	~~χ~~
~~P~~	~~D_t~~	~~$-D_k$~~	~~$F_{k\ell}$~~	~~$-F_{0\ell}$~~	~~χ (up to a phase)~~

(2) Symmetries

gauge invariance: use gauge-invariant/covariant combinations

$\vec{E}$, $\vec{B}$, $\frac{D_t}{c} = \frac{\partial_t}{c} + i\frac{e}{c}\varphi$ $\quad D_k = \partial_k + i\frac{e}{c}A_k$

Parity P and Time-reversal T

	iD_t	iD_k	E_k	B_k	$\bar{\chi}\chi$	$\bar{\chi}\sigma^k\chi$
P	iD_t	$-iD_k$	$-E_k$	B_k	$\bar{\chi}\chi$	$\bar{\chi}\sigma^k\chi$
T	$+iD_t$ $= -i(-D_t)$	$-iD_k$ $(-i)D_k$	$+E_k$	$-B_k$	$\bar{\chi}\chi$	$-\bar{\chi}\sigma^k\chi$

(up to phases)

Charge conjugation C ?

broken since $e^- \overset{C}{\longleftrightarrow} e^+$ and we do not need e^+.

(3) Power counting

It can be guessed from our exercise.

c appear only in the combinations

mc ,	$\frac{\partial_t}{c}$,	$\frac{e}{c}E_k$,	$\frac{e}{c}B_k$,	∂_k
l^{-1}	l^{-1}	l^{-2}	l^{-2}	l^{-1}

In bottom-up approach:

$$\mathcal{L}_{EFT} = \sum_n \xi_n(\alpha)\,\frac{O^{(n)}}{c^n} + \text{" }\gamma\text{ "}$$

$$O^{(n)} = \bar{\chi}\left[\quad l^{-1}\quad\right]\chi$$

Most general operator

$$\bar{\chi}\left[\left(\frac{D_t}{c}\right)^p (D_k)^q \left(\frac{e}{c}E_e\right)^r \left(\frac{e}{c}B_e\right)^m \frac{1}{(mc)^l}\right]\chi$$

with $\quad p, q, r, m, l \geq 0$ integers

[allowed operator $\bar{\chi}\, mc\, \chi$ can be removed, we will see how.
More convenient to have all $n \geq 0$]

Conditions:

dimension l^{-1}

$$-p-q-2r-2m+l=-1 \qquad (1)$$

$O(1/c^n)$:

$$p+r+m+l=n \qquad (2)$$

Parity:

$(q+r)$ is <u>even</u>

Adding (1) and (2)

$$\boxed{\begin{aligned} &-(q+r)-m+2l=(n-1)\\ &p+r+m+l=n\\ &(q+r)\ \text{even}\end{aligned}}$$

$n = 0$ $\quad p + r + m + l = 0 \rightarrow p = r = m = l = 0$

$-q = -1 \quad q = +1$

q even $\rightarrow$ no solution

$n = 1$ $\quad p + r + m + l = 1$

$-(q+r) - m + 2l = 0$

$(q+r)$ even

solution: $p = +1 \quad r = m = l = q = 0$

$$\boxed{\bar{\chi}\, i \frac{D_t}{c} \chi} \rightarrow \text{kinetic term for } \chi$$

$n = 2$ $\quad p + r + m + l = 2$

$-(q+r) - m + 2l = 1$

$(q+r)$ even

$l = 0 \quad -q - r - m = +1$ no solution

$l = 1 \quad p + r + m = 1$

$$\left.\begin{array}{l} (q+r) + m = 1 \\ (q+r) \text{ even} \end{array}\right\} \begin{array}{l} m = 1 \\ p = q = r = 0 \end{array}$$

$$\boxed{\bar{\chi}\, \sigma_k \frac{e}{c} B_k \frac{1}{mc} \chi}$$

anomalous magnetic moment of electron

$n = 1$ also $l = 1 \rightarrow p = r = m = 0$

$q = 2$

$$\boxed{\bar{\chi} \frac{D_k^2}{mc} \chi}$$

$n = 3$

$$-(q+r) - m + 2l = 2$$
$$p + r + m + l = 3$$
$$(q+r) \text{ even}$$

$l = 0$

$$-(q+r) - m = 2 \quad \to \quad \text{no solution}$$
$$p + r + m = 3$$
$$(q+r) \text{ even}$$

$l = 1$

$$-(q+r) - m = 0 \quad \to \quad q = r = m = 0$$
$$p + \cancel{l}\ 1 = 3 \qquad p = 2$$
$$(q+r) \text{ even} \qquad \text{o.k.}$$

$$\boxed{\bar{x} \; \frac{D_t^2}{m c^3} \; x}$$

we will see that this is redundant

$l = 2$

$$\left.\begin{array}{l} -(q+r) - m = -2 \\ p + r + m = +1 \\ (q+r) \text{ even} \end{array}\right\} \quad -q + p = -1$$

solutions

$p = 0$	$q = +1 = r$		$m = 0$
$p = 1$	$q = 2$	$r = 0 = m$	
~~$p = 2$~~	$q = 3$	no sol	

$p = 0$ $\quad \bar{x} \left[\text{"} D_k \frac{E_l}{c} \text{"} \cdot \frac{1}{m^2 c^2} \right] x$

$p = 1$ $\quad \bar{x} \left[\text{"} \frac{D_t}{c} \; D_k^2 \text{"} \frac{1}{m^2 c^2} \right] x$ $\quad$ redundant

$l = 3$ $\quad -(q+r) - m = -4 \quad \rightarrow \quad q = 4$

$\quad p + r + m = 0 \quad \rightarrow \quad p = r = m = 0$

$\quad (q+r)$ even

$$\bar{\chi}\left[\frac{\text{"}D_k^4\text{"}}{m^3 c^3}\right]\chi$$

$l = 4$ $\quad -(q+r) - m = -6$

$\quad p + r + m = -1 \quad$ (no solution)

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Taking into account also T we find the gauge-invariant, P and T invariant combinations:

$$\mathcal{L} = \bar{\chi}\, i \frac{D_t}{c} \chi + c_2\, \bar{\chi} \frac{D^2}{2mc} \chi$$

$$+ \; c_F \frac{e}{2mc^2} \bar{\chi}\, \vec{\sigma} \cdot \vec{B}\, \chi \qquad \text{(Fermi)}$$

$$+ \; c_D \frac{e}{8m^2c^3} \bar{\chi} (\vec{D}\cdot\vec{E} - \vec{E}\cdot\vec{D}) \chi \qquad \text{(Darwin)}$$

$$+ \; c_S \frac{ie}{8m^2c^3} \bar{\chi}\, \vec{\sigma} \cdot (\vec{D}\times\vec{E} - \vec{E}\times\vec{D}) \chi \qquad \text{(Spin-orbit)}$$

$$+ \; c_R \frac{\bar{\chi} (D^2)^2 \chi}{8m^3c^3} \qquad \text{(Relativistic)}$$

Parity and Time reversal in NRQED

hermitian combinations are:

	iD_t/c	iD_k	E_k/c	B_k/c	$\bar{\chi}\chi$	$\bar{\chi}\sigma^k\chi$
[·]	ℓ^{-1}	ℓ^{-1}	ℓ^{-2}	ℓ^{-2}	·	·
P	+	−	−	+	+	+
T	$(-i)(-D_t/c)$	$(-i)D_k$	+	−	+	−

c.c. since T is antilinear

→ $\bar{\chi}\ iD_t\ \chi$ o.k.

$+ \quad +$ P

$+ \quad +$ T

$\bar{\chi}\ D_k^2\ \chi$ o.k.

$+ \quad +$ P

$+ \quad +$ T

$\bar{\chi}\ D_t^2\ \chi$ o.k.

$+ \quad +$ P

$+ \quad +$ T

$\bar{\chi}\ D^4\ \chi$ o.k.

$+ \quad +$ P

$+ \quad +$ T

$\bar{\chi}\ D\cdot E\ \chi$ o.k.

$+ \quad - \quad -$ P

$+ \quad + \quad +$ T

$\bar{\chi}\ i\vec{\sigma}\cdot\vec{D}\times\vec{E}\ \chi$ o.k.

$+ \quad - \quad -$ P

$- \quad - \quad +$ T

To have the kinetic term at LO, i.e. c-independent we can redefine

$$\chi \to \sqrt{c}\,\chi$$

and we end up with:

$$\mathcal{L}_{NRQED} = \bar{\chi}\, i D_t \chi + c_2\, \bar{\chi} \frac{D^2}{2m} \chi$$
$$+ c_F \frac{e}{2mc} \bar{\chi}\, \vec{\sigma} \cdot \vec{B}\, \chi$$
$$+ c_D \frac{e}{8m^2c^2} \bar{\chi} (\vec{D}\cdot\vec{E} - \vec{E}\cdot\vec{D}) \chi$$
$$+ c_S \frac{ie}{8m^2c^2} \bar{\chi} (\vec{\sigma} \cdot \vec{D}\times\vec{E} - \vec{E}\times\vec{D}) \chi$$
$$+ c_R \frac{\bar{\chi}\, D^4 \chi}{8m^3c^2} + O\left(\frac{1}{c^3}\right)$$

Notice that the expansion is in $(1/c)$ not in $(1/m)$. Indeed, when computing the fine structure of the hydrogen atom the 3 terms D, S, R contribute at the same order although $D \sim S \sim m^{-2}$ and $R \sim m^{-3}$.

Next step:

The coefficients c_2, c_F, c_D, c_S, c_R are of the type:

$$c_i = c_i(\alpha)$$

→ Match the EFT to ~~the~~ QED to LO in α. We will see that

$$C_i(\alpha) = 1 + O(\alpha)$$

We have

$$\mathcal{L} = \bar{\Psi}\left(i\gamma^0 \frac{D_t}{c} + i\gamma^k D_k - mc\right)\Psi + \ldots$$

1st step

Most of the e^- energy is mc^2. We can get rid of it by redefining

$$\Psi = e^{-imc^2 t}\tilde{\Psi}$$

This is like changing the zero of energies.

$$iD_t\Psi = e^{-imc^2t}\left(iD_t\tilde{\Psi} + mc^2\tilde{\Psi}\right)$$

that gives

$$\mathcal{L} = \bar{\tilde{\Psi}}\left[i\gamma^0\frac{D_t}{c} + i\gamma^k D_k + mc(\gamma^0 - \mathbb{1})\right]\tilde{\Psi}$$

$$(\gamma^0 - \mathbb{1}) = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}$$

$$mc(\gamma^0 - \mathbb{1})\begin{pmatrix}\tilde{\chi} \\ \tilde{\bar{\eta}}\end{pmatrix} = \begin{pmatrix} 0 \\ -2mc\,\tilde{\bar{\eta}}\end{pmatrix}$$

Now the e^+ has "reference energy" $-2mc^2$

Attention: here $\chi \to \tilde{\varphi}$
$\eta \to \tilde{\chi}$

2nd step

Write the classical e.o.m. and remove the field $\tilde{\eta}$ (e^+) by substituting the solution $\hat{\eta} = \tilde{\eta}(\tilde{\varphi})$ into $\mathcal{L}$.

$$\begin{bmatrix} i\dfrac{D_t}{c} & i\sigma^k D_k \\ -i\sigma^k D_k & -i\dfrac{D_t}{c} - 2mc \end{bmatrix} \begin{bmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{bmatrix} = 0$$

$$\begin{cases} i\dfrac{D_t}{c}\tilde{\varphi} + i\sigma^k D_k \tilde{\chi} = 0 \\ -i\sigma^k D_k \tilde{\varphi} - i\dfrac{D_t}{c}\tilde{\chi} - 2mc\tilde{\chi} = 0 \end{cases}$$

This system can be solved iteratively by expanding in $1/c$.
Take the 2nd equation at LO

$$-i\sigma^k D_k \tilde{\varphi} - 2mc\tilde{\chi} = 0 \quad \to \quad \tilde{\chi} = \frac{-i\sigma^k D_k}{2mc}\tilde{\varphi} \equiv \frac{\tilde{\chi}_1}{c}$$

Now consider 1st correction: $\tilde{\chi} = \dfrac{\tilde{\chi}_1}{c} + \dfrac{\tilde{\chi}_2}{c^2}$
and show that $\tilde{\chi}_2 = 0$
2nd correction $\tilde{\chi} = \dfrac{\tilde{\chi}_1}{c} + \dfrac{\tilde{\chi}_3}{c^3}$

$$\cancel{-i\sigma^k D_k \tilde{\varphi}} - i\frac{D_t}{c}\frac{\tilde{\chi}_1}{c} - \cancel{2mc\frac{\tilde{\chi}_1}{c}} - \frac{2m}{c^2}\tilde{\chi}_3 = 0$$

$$-iD_t\tilde{\chi}_1 = 2m\tilde{\chi}_3 \qquad \tilde{\chi}_3 = \frac{-iD_t}{2m}\tilde{\chi}_1$$

$$\tilde{\chi} = \left[\frac{-i\sigma^k D_k}{2mc} - \frac{iD_t}{2mc^2}\cdot\frac{-i\sigma^k D_k}{2mc} + \dots\right]\tilde{\varphi}$$

We can go on and formally obtain

$$\boxed{\tilde{\chi} = \frac{1}{\left(1 + \frac{i D_t}{2mc^2}\right)} \cdot \frac{-i\sigma^k D_k}{2mc} \tilde{\varphi}}$$

The 1^{st} eq. becomes the e.o.m. for the field $\tilde{\varphi}$

$$i \frac{D_t}{c} \tilde{\varphi} + i\sigma^k D_k \frac{1}{\left(1 + i\frac{D_t}{2mc^2}\right)} \cdot \frac{-i\sigma^l D_l}{2mc} \tilde{\varphi} = 0$$

Formally this e.o.m. can be derived from the Lagrangian (set $\tilde{\varphi} = \sqrt{c}\, \psi$)

$$\mathcal{L} = \bar{\Psi} \left[i D_t + \sigma^k D_k \frac{1}{\left(1 + i\frac{D_t}{2mc^2}\right)} \cdot \frac{\sigma^l D_l}{2m} \right] \Psi$$

This is our "wilsonian" tree-level effective action. It is non local, but it can be expanded in powers of $1/c^2$ becoming a series of local terms:

$$\mathcal{L} = \bar{\Psi} \left[i D_t + \sigma^k D_k \left(1 - i\frac{D_t}{2mc^2}\right) \frac{\sigma^l D_l}{2m} + \dots \right] \Psi$$

$$= \bar{\Psi} \left[i D_t + \frac{\sigma^k D_k \sigma^l D_l}{2m} - \frac{i}{4m^2c^2} \sigma^k D_k D_t \sigma^l D_l + \dots \right] \Psi$$

As we will see tomorrow:

$$\boxed{\sigma^k D_k \sigma^l D_l = D^2 + \frac{e}{c}\vec{\sigma}\cdot\vec{B}}$$

which fixes $C_2 = C_F = 1$

The next term requires some manipulation

$$\sigma^k D_k D_t \sigma^l D_l = A D_t A \qquad A \equiv \sigma^k D_k$$

$$= \frac{1}{2} A[D_t, A] - \frac{1}{2}[D_t A, A] + \frac{1}{2} D_t A^2 + \frac{1}{2} A^2 D_t$$

$$A[D_t, A] = ie\, D_k E^k - e\, \epsilon^{lij} \sigma^l D_i E^j$$
$$= ie\, \vec{D}\cdot\vec{E} - e\, \vec{\sigma}\cdot\vec{D}\times\vec{E}$$

$$[D_t A, A] = ie\, E^k D_k - e\,\vec{\sigma}\cdot\vec{E}\times\vec{D} +$$

$$\sigma^k D_k D_t \sigma^l D_l = \frac{i}{2} e(\vec{D}\cdot\vec{E} - \vec{E}\cdot\vec{D})$$
$$- \frac{e}{2}\vec{\sigma}\cdot(\vec{D}\times\vec{E} - \vec{E}\times\vec{D}) - \frac{e}{c} D_t \vec{\sigma}\cdot\vec{B}$$
$$+ \frac{1}{2} D_t A^2 + \frac{1}{2} A^2 D_t$$

Total Lagrangian:

$$\mathcal{L} = \bar{\Psi}\Big[i D_t + \frac{D^2}{2m} + \frac{e}{2mc}\vec{\sigma}\cdot\vec{B}$$
$$+ \frac{1}{8m^2c^2} e(\vec{D}\cdot\vec{E} - \vec{E}\cdot\vec{D})$$
$$+ \frac{ie}{8m^2c^2}\vec{\sigma}\cdot(\vec{D}\times\vec{E} - \vec{E}\times\vec{D}) +$$
$$+ \frac{(-i)}{4m^2c^2}\left[\frac{1}{2} D_t A^2 + \frac{1}{2} A^2 D_t\right]\Big]\Psi$$

We can get rid of the last line by redefining

$$\Psi \to \left(1 + \frac{1}{8m^2c^2} A^2\right)\Psi$$

$$\bar{\Psi} \to \bar{\Psi}\left(1 + \frac{1}{8m^2c^2} \overleftarrow{A}^2\right)$$

Keeping only terms up to $(1/c^2)$ we have

$$\bar{\Psi}\left(1 + \frac{1}{8m^2c^2}\overleftarrow{A}^2\right)\left(iD_t + \frac{D^2}{2m}\right)\left(1 + \frac{1}{8m^2c^2}A^2\right)\Psi + \dots$$

$$= \bar{\Psi}\, iD_t \Psi + \frac{1}{8m^2c^2}\bar{\Psi}\left(\overleftarrow{A}^2 iD_t + iD_t A^2\right)\Psi$$ ← this eliminates the last line

$$+ \bar{\Psi}\frac{D^2}{2m}\Psi + \frac{\bar{\Psi}}{16m^3c^2}\left(\overleftarrow{A}^2 D^2 + D^2 A^2\right)\Psi + \dots$$

$$= \bar{\Psi}\left(iD_t + \frac{D^2}{2m}\right)\Psi + \frac{1}{16m^3c^2}\bar{\Psi}\left(\underbrace{A^2D^2 + D^2A^2}_{=2D^4 + \text{higher orders}}\right)\Psi + \dots$$

Putting everything together:

$$\mathcal{L} = \bar{\Psi}\Big[iD_t + \frac{D^2}{2m} + \frac{e}{2mc}\vec{\sigma}\cdot\vec{B}$$

$$+ \frac{1}{8m^2c^2} e(\vec{D}\cdot\vec{E} - \vec{E}\cdot\vec{D})$$

$$+ \frac{ie}{8m^2c^2}\vec{\sigma}\cdot(\vec{D}\times\vec{E} - \vec{E}\times\vec{D})$$

$$+ \frac{1}{8m^3c^2} D^4 + \dots \Big]\Psi$$

how to eliminate the term

$$\frac{\xi}{mc^2}\,\bar{\chi}\,D_t^2\,\chi \qquad \text{from } \mathcal{L}_{NRQED}$$

consider the field redefinition

$$\boxed{\begin{aligned} &\chi \longrightarrow \left(1+\frac{\vec{\Delta}}{c^2}\right)\chi \\ &\vec{\Delta} \equiv \frac{\xi}{2m}\left(iD_t - \frac{D^2}{2m}\right) \end{aligned}}$$

$$\mathcal{L}_{NRQED} = \bar{\chi}\left(1+\frac{\overleftarrow{\Delta}^*}{c^2}\right)\left(iD_t + \frac{D^2}{2m}\right)\left(1+\frac{\vec{\Delta}}{c^2}\right)\chi + \\ + \cdots + \frac{\xi}{mc^2}\,\bar{\chi}\,D_t^2\,\chi + O\left(\frac{1}{c^3}\right)$$

$$= \bar{\chi}\left(iD_t + \frac{D^2}{2m}\right)\chi \equiv LO$$

$$+ \bar{\chi}\left[\frac{\overleftarrow{\Delta}^*}{c^2}\left(iD_t + \frac{D^2}{2m}\right) + \left(iD_t + \frac{D^2}{2m}\right)\frac{\vec{\Delta}}{c^2}\right]\chi + \cdots$$

$$+ \frac{\xi}{mc^2}\,\bar{\chi}\,D_t^2\,\chi$$

$$= LO + \frac{\xi}{2mc^2}\,\bar{\chi}\left[\left(iD_t - \frac{D^2}{2m}\right)\left(iD_t + \frac{D^2}{2m}\right) + \left(iD_t + \frac{D^2}{2m}\right)\left(iD_t - \frac{D^2}{2m}\right)\right]\chi$$

$$+ \frac{\xi}{mc^2}\,\bar{\chi}\,D_t^2\,\chi$$

$$\mathcal{L}_{NRQED} = LO + \frac{\xi}{2mc^2}\,\bar{\chi}\left[\left(-\cancel{D_t^2} - \frac{D^4}{4m^2}\right)\times 2\right]\chi$$

$$+ \cancel{\frac{\xi}{mc^2}\,\bar{\chi}\,D_t^2\,\chi} \quad + \dots$$

$$= LO + \frac{(-\xi)}{4m^3c^2}\,\bar{\chi}\,D^4\chi \quad + \dots$$

but we already have $\bar{\chi}D^4\chi$ in the list.

Notice that the overall effect is equivalent to the use of e.o.m.

$$\left(iD_t + \frac{D^2}{2m}\right)\chi = O\left(\frac{1}{c}\right)$$

$$\frac{\xi}{mc^2}\,\bar{\chi}\,D_t^2\,\chi = -\frac{\xi}{mc^2}\,\bar{\chi}(-i\overleftarrow{D}_t)(i\overrightarrow{D}_t)\chi$$

$$= -\frac{\xi}{mc^2}\,\bar{\chi}\left(-\frac{\overleftarrow{D}^2}{2m}\right)\left(-\frac{D^2}{2m}\right)\chi \quad + \dots$$

$$= -\frac{\xi}{4m^3c^2}\,\bar{\chi}\,D^4\chi \quad + \dots$$