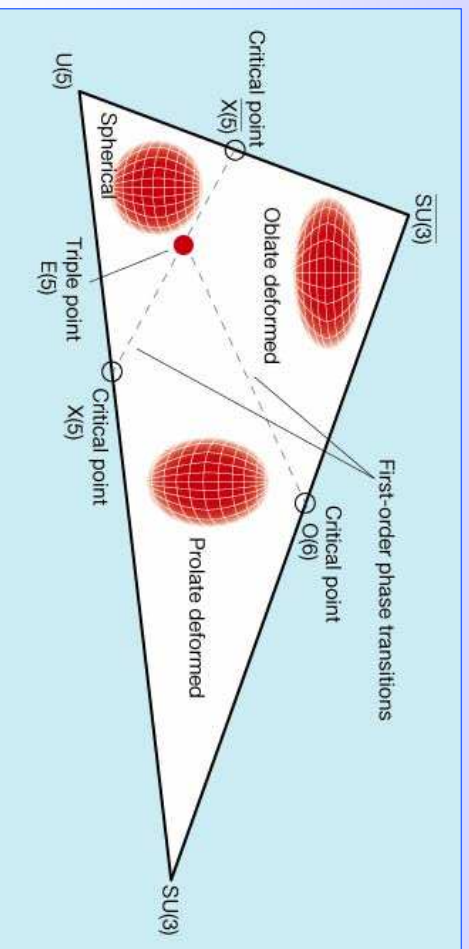


# Algebraic approaches and dynamical symmetries in quantum many-body systems

L. Fortunato – Padova Univ. (Italy)



Mini-cycle of lectures given at

Oslo University Jan. – Feb. 2009

## Programme for the lectures' cycle

- Introduction to Lie algebras, properties , classification, Casimir operators (REPETTA JUVANT)
- Dynamical symmetries and spectrum generating algebra. Rigid rotor ( $SO(3)$ ). Examples in Particle, Atomic and Molecular physics (fun for everybody!)
- IBM-1:  $U(6)$  algebra, branching and subalgebra chains, Casten's triangle, examples of spectra and  $B(E2)$ , selection rules.
- IBM-2, etc.; IBFM and fermionic models, supersymmetry in nuclear physics, example (Au-Pt) symmetries
- **Seminar**: Shape phase transitions and critical point symmetries

# 1 – Mathematical introduction of Lie algebras



Sophus Lie  
norwegian  
mathematician

- Lie algebras, subalgebras and Cartan classification
- Commutator and structure constants
- Representations and reducibility
- Rank, order, Casimir operators
- Spectrum Generating Algebra
- Dynamical symmetries
- Rotor, angular momentum and  $SO(3)$

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## Lie Algebra $G = \{g_1, g_2, \dots, g_n\}$ :

Vetorial space of dimension  $n$ , the elements of which are operators (infinitesimal), endowed with an operation, called Lie product or commutator  $[.., ..]$  such that:

- $[g_i, g_j] = \sum c_{ij}^k g_k \quad \forall g_k \in G \quad \rightarrow$  Closure  $[g_i, g_j]$
- $[a, b] = -[b, a] \quad \forall a, b \in G \quad \rightarrow$  Antisymmetry
- $[\alpha a + \beta b, c] = \alpha [a, c] + \beta [b, c] \quad \rightarrow$  Bilinear
- $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \quad \rightarrow$  id. Jacobi

$c_{ij}^k$  are the structure constants



They completely specify the Lie algebra

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## Abelian Algebra, Invariant Subalgebra

An algebra is **Abelian**, or commutative, if

$$[a, b] = 0 \quad \forall a, b \in G \quad \Leftrightarrow \text{null struct.const.} \quad c^k_{ij} = 0$$

A **Subalgebra**  $G'$  of a given Lie algebra  $G$  is a subset  $G \supset G'$  of the elements of  $G$  that is also an algebra on its own:

- it is said **proper**, if at least one element  $G$  is not contained in  $G'$
- it is said **invariant**, if  $[a, b] \in G' \quad \forall a \in G' \quad \forall b \in G$  (Invariant Subalgebra = **Ideal**)

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## Direct Sum of Lie Algebras

$G$  is the **direct sum** of two Lie algs.  $G = G_1 \oplus G_2$ , if:

- the vect.sp.  $G$  is direct sum of two spaces  $G_1, G_2$  or better if  $a \in G_1$  and  $b \in G_2$  than  $c \in G$  can be written in a unique way as  $c = a + b$ . Said another way  $G_1, G_2$  have in common only the null element (i.e. they are orthogonal vector spaces)

- $\forall a \in G_1$  and  $\forall b \in G_2$  it must be that  $[a, b] = 0$

And also

$$B = \{\underbrace{a_1, a_2, \dots, a_I}_{G_1}, \underbrace{a_{I+1}, \dots, a_n}_{G_2}\} \text{ is a basis of } G$$

From which you get  $\dim(G) = \dim(G_1) + \dim(G_2)$

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## Simple, Semisimple, etc.

An algebra is called **simple** if it is not abelian and it has no proper invariant subalgebras.

An algebra is called **semisimple** if it doesn't have any abelian invariant subalgebra.

Coroll.: simple  $\Rightarrow$  semisimple

**Cartan Criterion** : An algebra  $G$  is semisimple if and only if its Killing form (the metric tensor) is non-degenerate,  $\det(g) \neq 0$

$$g_{\mu\nu} = C^\sigma_{\mu\rho} C^\rho_{\nu\sigma}$$

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## Representations of a Lie algebra

Suppose that  $\forall a \in G$  there exists a matrix  $d \times d$ ,  $M(a)$  such that :

- $M(\alpha a + \beta b) = \alpha M(a) + \beta M(b)$
- $M([a, b]) = [M(a), M(b)]$

Then these matrices form a **representation** (matricial in this specific case)  $d$ -dimensional of  $G$ .

If  $M'(a) = S^{-1} \cdot M(a) \cdot S$

with  $S$ =matrix  $d \times d$  non-singular, then  $M, M'$  are **equivalent representations**

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# Reducibility

A given representation  $M(a)$  of the algebra  $G$  is **reducible** if it is equivalent a representation that might be written,  $\forall a \in G$ , in the form:

$$\begin{bmatrix} M_{11}(a) & M_{12}(a) \\ 0 & M_{22}(a) \end{bmatrix}$$

Viceversa it is called **irreducible** if it's not reducible.

It is instead **completely reducible** if, for example,

$$\begin{bmatrix} M_{11}(a) & 0 & 0 & \dots & 0 \\ 0 & M_{22}(a) & 0 & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & M_{nn}(a) \end{bmatrix}$$

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# Cartan Classification

All semisimple Lie algebras have been classified:



Elie Joseph Cartan  
French mat.

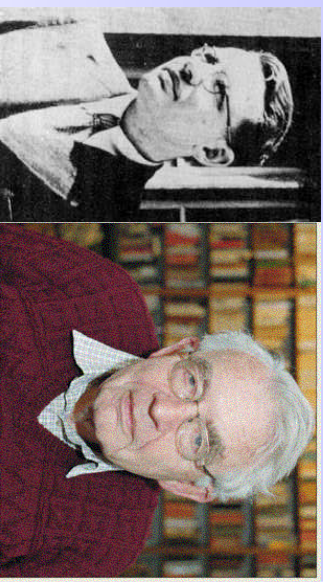
TABLE 1. Cartan classification of classical Lie algebras.

Name	Label	Cartan	Order( $r$ )	Rank( $\ell$ )
Special Unitary	$su(n)$	$A_\ell$	$n^2 - 1$	$n - 1$
(Special) Orthogonal	$so(n), n = \text{odd}$	$B_\ell$	$n(n-1)/2$	$(n-1)/2$
Symplectic	$sp(n), n = \text{even}$	$C_\ell$	$n(n+1)/2$	$n/2$
(Special) Orthogonal	$so(n), n = \text{even}$	$D_\ell$	$n(n-1)/2$	$n/2$
Exceptional	$G_2$	$G_2$	14	2
	$F_4$	$F_4$	52	4
	$E_6$	$E_6$	78	6
	$E_7$	$E_7$	133	7
	$E_8$	$E_8$	248	8
Non-semisimple				
Unitary	$u(n)$		$n^2$	$n$

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# Casimir Operators and Rank

For each algebra one can construct a set of operators, called **Casimir operators or invariants**,  $C$ , such that  $[C, X_i] = 0, \forall X_i \in G$



Hendrik Casimir (1909-2000) dutch physicist

The number of independent invariants is called **rank** of the algebra, namely  $\#C$ .

The **order** is the number of operators (generators) that form the algebra, namely  $\#X_i \in G$ .

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Example: Angular Momentum  $\rightarrow \mathfrak{so}(3)$

$\mathbf{L} = \mathbf{r} \times \mathbf{p} \rightarrow L_x, L_y, L_z$  are generators of  $\mathfrak{so}(3)$

$[L_x, L_y] = iL_z$  +cyclic permutations over indices

$$C[\mathfrak{so}(3)] = \mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$$

is the quadratic Casimir operator (rank 1)

$$[C, L_i] = 0, \forall L_i \in \mathfrak{so}(3)$$



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... continues ...

One must specify the action of the generators on a given orthonormal basis  $|LM\rangle$

$$L_{\pm}|LM\rangle = [L(L\mp M)(L\pm M+1)]^{1/2}|LM\pm 1\rangle \quad L_{\pm} = L_x \pm iL_y$$

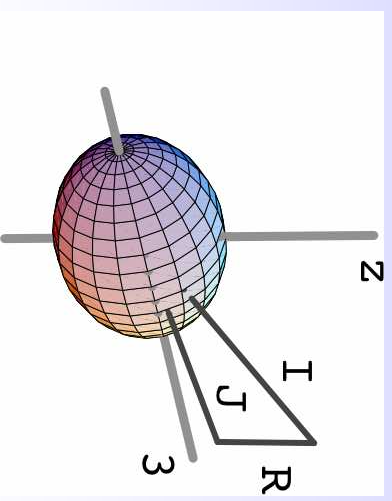
$$L_z|LM\rangle = M|LM\rangle$$

$$L^2|LM\rangle = L(L+1)|LM\rangle$$

Application in the **rigid rotor**:

$$H = k L^2 \rightarrow E = k L(L+1)$$

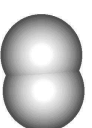
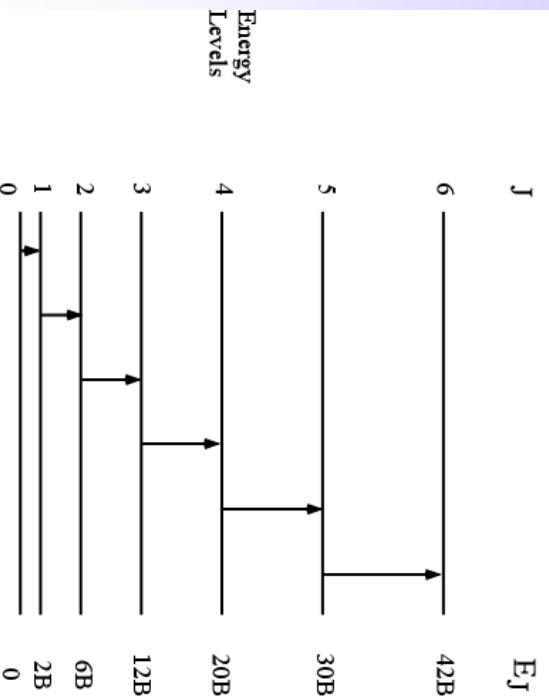
with  $k = (h^2/2I)$



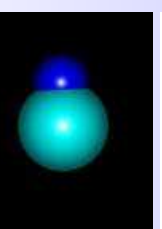
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... quantum rotors

### ROTATIONAL ENERGY LEVELS AND TRANSITIONS OF A DIATOMIC MOLECULE



$H_2$



HCl,  
CO, etc.

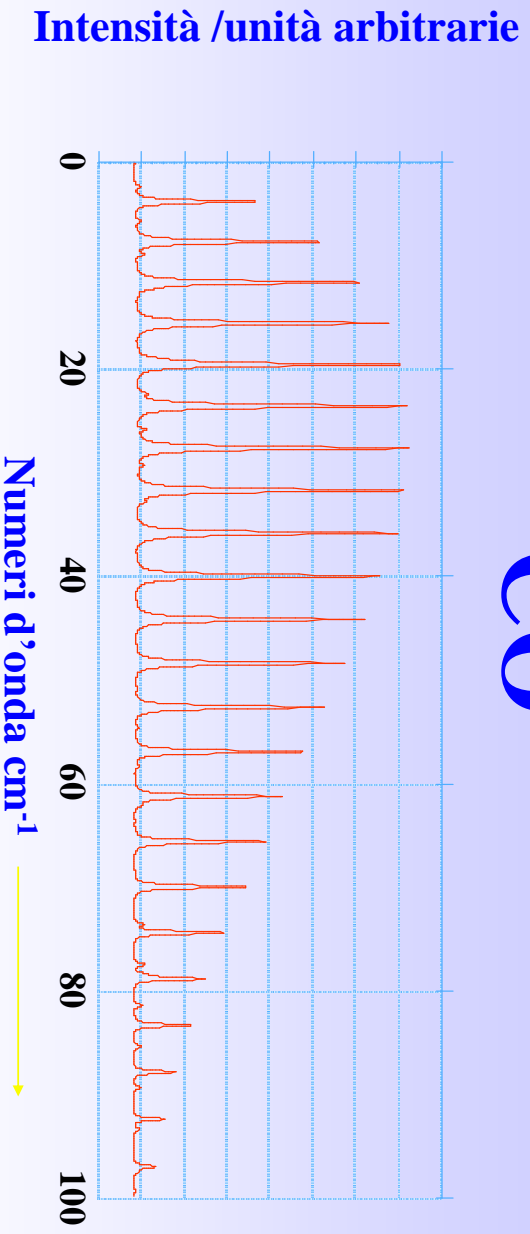
$$E_J = B J(J+1)$$

$$\Delta E_J = 2B(J+1)$$

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# CO



$$E_J = B J(J+1)$$

$$\Delta E_J = 2B(J+1) \propto \nu$$

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## Spectrum Generating Algebra (SGA)

When, in general, one can write an hamiltonian

$$H = E_0 + \sum C_k X_k + \sum C'_k X'_k + \dots \quad \text{with} \quad X_k \in G$$

as a polynomial in the elements of an algebra

then  $G$  is called **spectrum generating algebra (SGA)** for  $H$ , because it is always possible to diagonalize (numerically)  $H$  in the ONC basis labelled by all the quantum numbers of a **Complete Set of Commuting Operators (CSCO)** dof any of the possible chains of subalgebras of  $G \supset G' \supset G'' \supset \dots$

Once the *action of the*  $X_k$  on  $|\alpha\rangle$  is given, then one can calculate the matrix elements  $\langle \alpha | H | \alpha' \rangle$

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## Example of SGA

The non-compact algebra  $SO(2,1)$  can be realized with the following differential operators:

$$Z_1 = \mathbf{p}^2 \quad Z_2 = \mathbf{r}^2 \quad Z_3 = k(\mathbf{r} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{r})$$

These operators close under commutation with the structure constants typical of the  $SO(2,1)$  algebra.

Once the action of the  $Z_k$  on some  $|\alpha\rangle$  is given, then one can calculate the matrix elements  $\langle \alpha | H | \alpha' \rangle$  of hamiltonians, for example the harmonic oscillator :

$$H = \frac{1}{2} \mathbf{p}^2 + \frac{1}{2} \mathbf{r}^2$$

(a part from some constants)

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## Dynamical Symmetry

In some cases

$$H = E_0 + \sum C_k X_k + \sum C'_k X'_k + \dots \quad \text{with} \quad X_k \in G$$

we have only some terms that correspond to invariant operators of the algebras in the chain:

$$G \supset G' \supset G'' \supset \dots \quad \text{chain of subalgebras}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$C \quad C' \quad C'' \quad \text{(one or more for each subalgebra)}$$

$$H = E_0 + aC + a'C' + a''C'' + \dots$$

in these cases we speak of a **dynamical symmetry** and we have

$$E = E_0 + a \langle C \rangle + a' \langle C' \rangle + a'' \langle C'' \rangle + \dots$$

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## Consequences of a DS -1

1) All the states are soluble and we have analytic expressions for energy and other observables

$$E = E_0 + a\langle C \rangle + a'\langle C \rangle + a''\langle C^2 \rangle + \dots$$

2) All the states are characterized by quantum numbers that "label" the irreducible representations (IRREPS) of the chain of subalgebras

$$|\alpha_1 \alpha_2 \dots \alpha_n\rangle$$

3) The structure of the wavefunctions is dictated by symmetry and it's independent from the details of the hamiltonian

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## Consequences of DS -2

Assume that H commutes with a set of operators that form a given Lie algebra:

$$\forall g_i \in G: [H, g_i] = 0$$

If  $|\gamma\rangle$  is eigenstate of H, then

$$H|\gamma\rangle = E|\gamma\rangle \Rightarrow H g_i |\gamma\rangle = g_i H |\gamma\rangle = E g_i |\gamma\rangle$$

also  $g_i |\gamma\rangle$  is an eigenstate of H and we have **degeneration**.

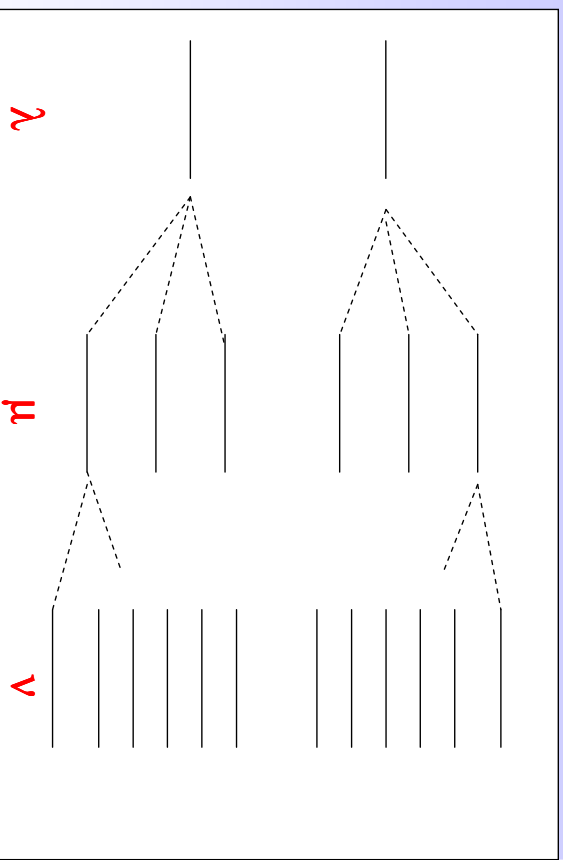
At the very origin of degeneration there is a conserved quantity, an invariant, that is the Casimir operator of some group.

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# Multiplets, degeneration and splitting.

$$\mathcal{B} = \{ | \lambda \mu \nu \rangle \}$$

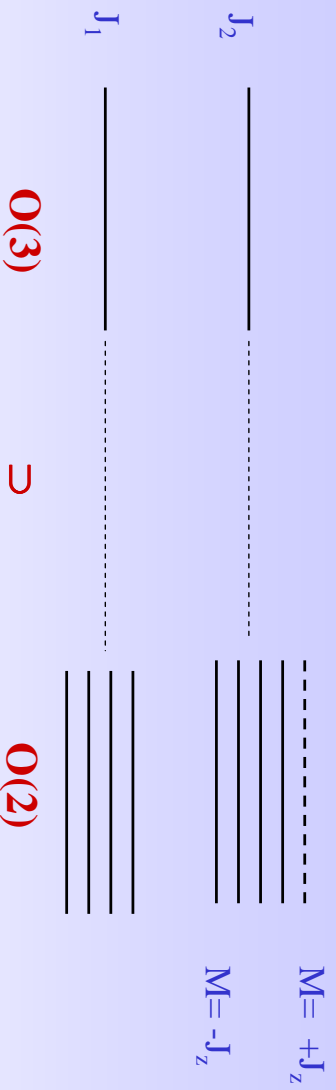
ONC basis states



The dynamical symmetry splits, but do not admix the states of a basis!

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## Magnetic Degeneration

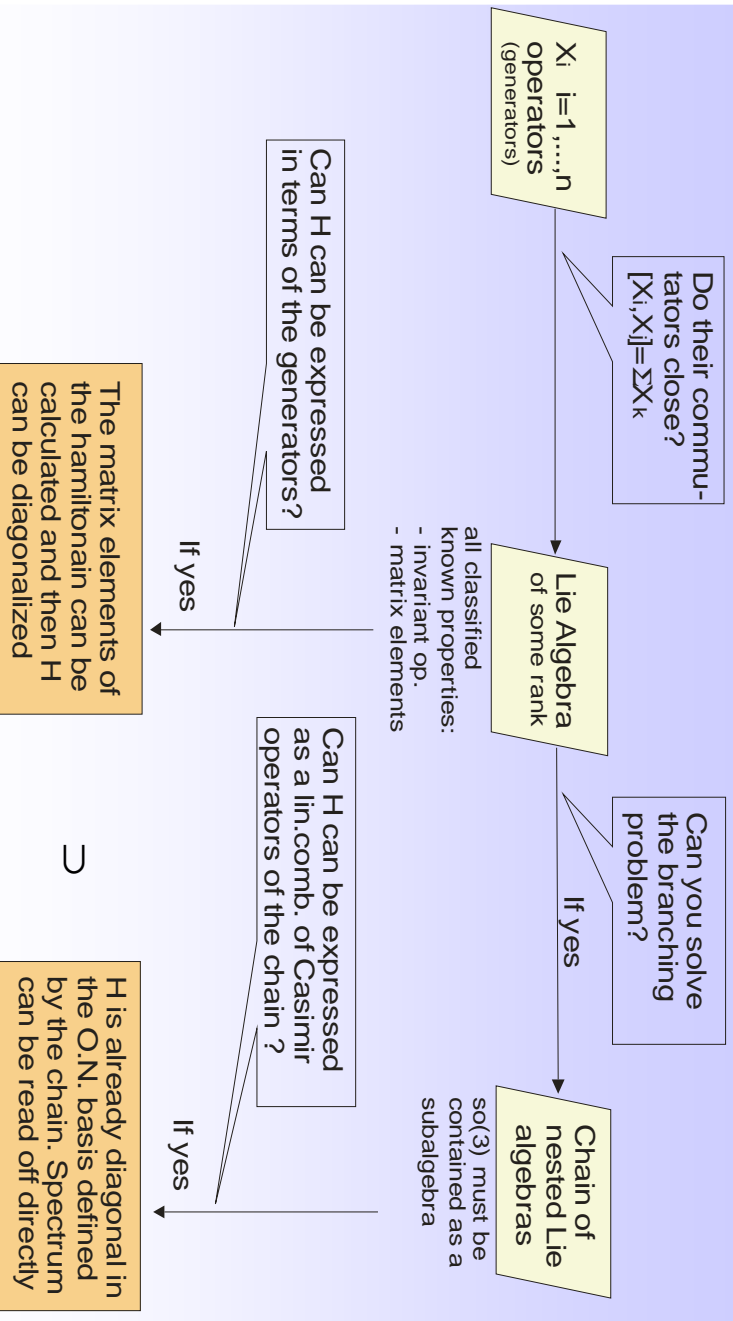


States with a definite total angular momentum contain a **multiplet** of substates with different third component component: these are called **magnetic substates** because can be separated with a magnetic field (Zeeman effect)

Said another way, the magnetic interaction  $-\mu B$  **breaks** the symmetry of the hamiltonian

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# Not easy to digest...let's go through it once again



## Spectrum generating algebra

## Dynamical Symmetry

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Example: hydrogen atom  $\rightarrow so(4)$

$$H = -\frac{p^2}{2M} - \frac{\alpha}{r}$$

$$H\Psi_{nlm}(r, \theta, \varphi) = -\frac{M\alpha^2}{2\hbar^2 n^2} \Psi_{nlm}(r, \theta, \varphi)$$

with  $n = 1, 2, \dots; l = 0, 1, \dots, n-1; m = -l, \dots, +l$

The states of the hamiltonian of the H atom are clearly invariant with respect to  $SO(3)$ , but...

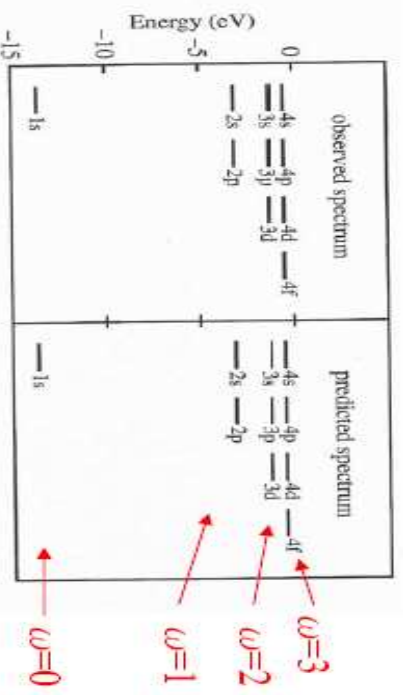
## Classification scheme

$$SO(4) \supset SO(3) \supset SO(2)$$

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ \omega & l & m \end{matrix}$$

$$m = \omega + 1 \quad \omega = 0, 1, \dots, \infty$$

$$l = \omega, \omega - 1, \dots, 1, 0$$



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## Further degeneration

The spectrum shows a further degeneration in  $\ell$ .  
Where does it come from ?

Degeneration  $\Rightarrow$  Conserved Quantity

Runge-Lenz vector:

$$\mathbf{A} = \frac{\mathbf{p} \wedge \mathbf{L}}{M} - \alpha \frac{\mathbf{r}}{r}, \quad \mathbf{A}^2 = \frac{2H}{M} \mathbf{L}^2 + \alpha^2$$

$$[L_j, L_k] = i\hbar \epsilon_{jkl} L_l, \quad j, k, l = x, y, z$$

$$[L_j, A_k] = i\hbar \epsilon_{jkl} A_l, \quad j, k, l = x, y, z$$

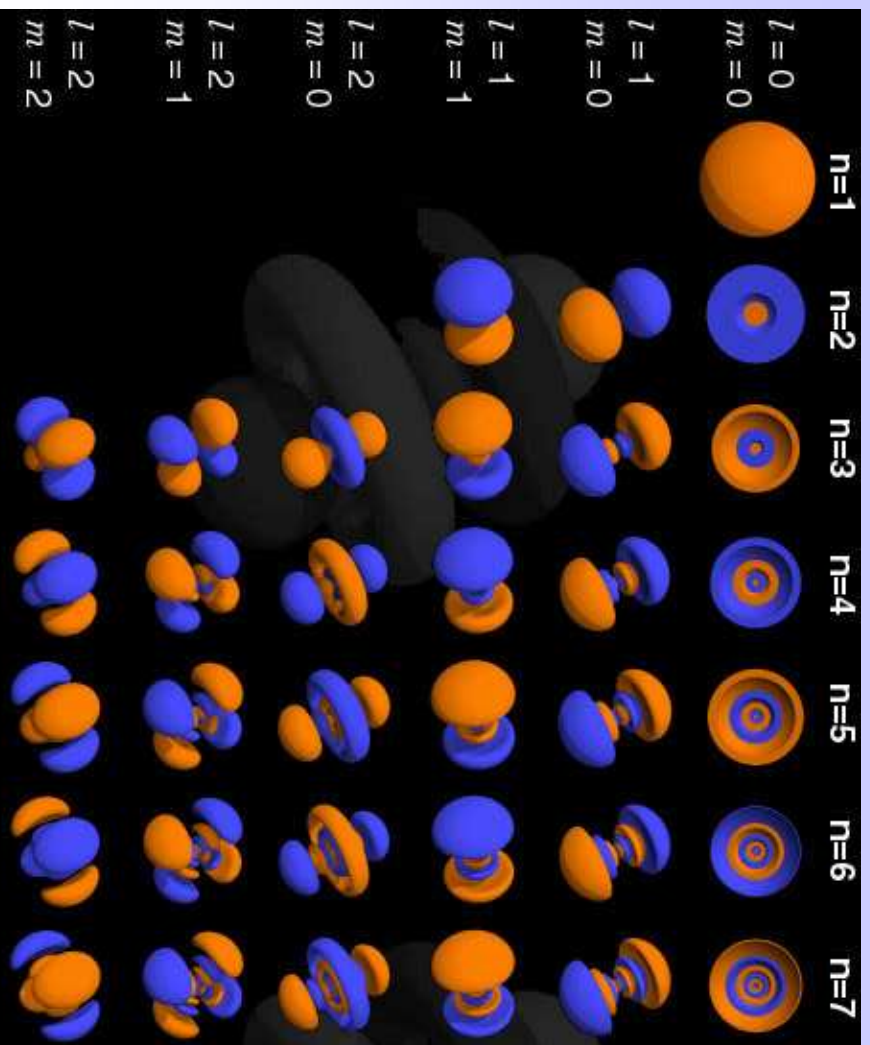
$$[A_j, A_k] = -i\hbar \epsilon_{jkl} \frac{2H}{M} L_l, \quad j, k, l = x, y, z$$

$$[H, A] = 0$$

$\Rightarrow$  there's a larger symm.  
group that contains both  
L and A

$$so(4) \approx so(3) \oplus so(3)$$

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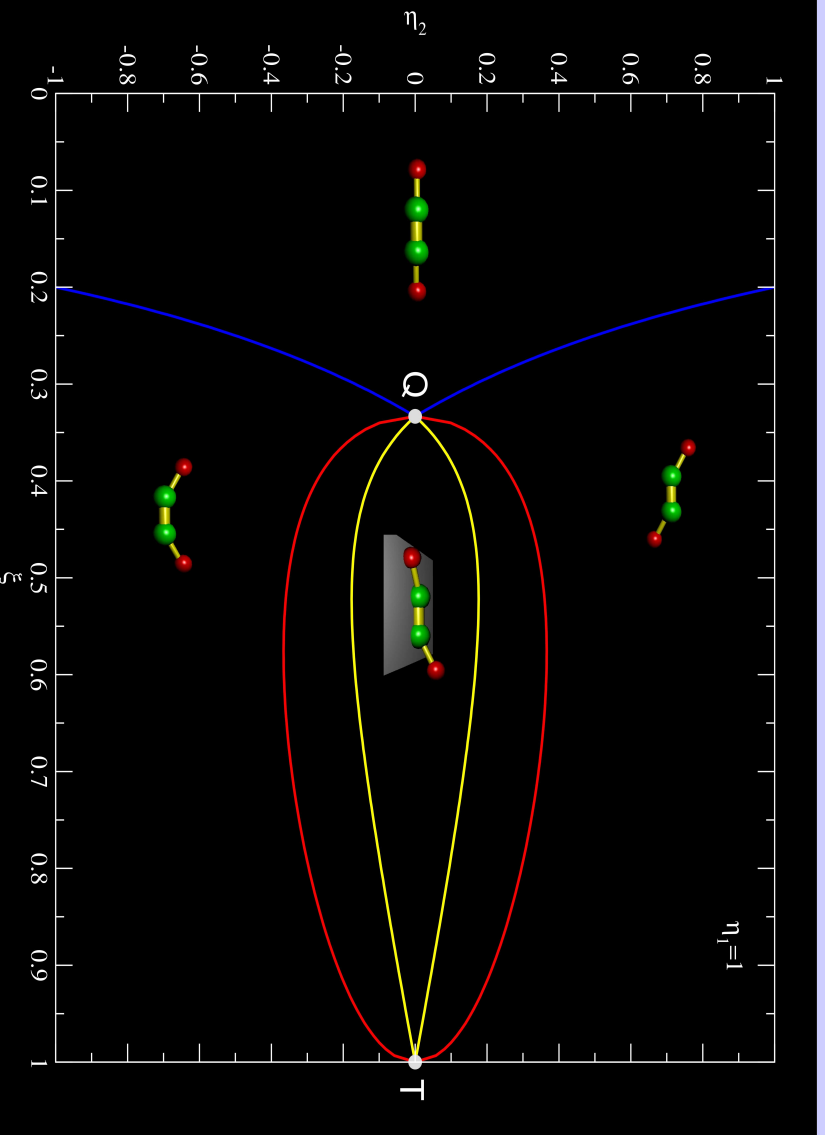
## Other important dynamical Symmetries

- **Hydrogen atom** (Pauli, 1926)
- **Isospin symmetry** (Heisenberg, 1932)
- **Spin-isospin symmetry** (Wigner, 1937)
- **Pairing, seniority** (Racah, 1943)
- **Elliott model** (Elliott, 1958)
- **Flavor symmetry** (Gell-Mann, Ne'eman, 1962)
- **Interacting boson model** (Arima, Iachello, 1974)
- **Nuclear supersymmetry** (Iachello, 1980)

From R.Bijker

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## Fisica Molecolare: acetilene $C_2H_2$



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# "Flavour" SU(3)

Gell-mann and Ne'eman have suggested that the internal degrees of freedom of hadrons can be described by an SU(3) dynamical symmetry (flavour)

$$\left. \begin{array}{ccccc} \text{SU}(3) & \supset & \text{SU}_I(2) & \otimes & \text{U}_Y(1) & \supset & \text{O}_I(2) & \otimes & \text{U}_Y(1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ (\lambda, \mu) & & I & & Y & & I_3 & & \end{array} \right\}$$

We have the Gellmann-Okubo mass formula

(Mass replaces energy in the relativistic formalism)

$$M=M_0+a\mathcal{C}_1(U_Y(1))+b[\mathcal{C}_2(\text{SU}_I(2))-\frac{1}{4}\mathcal{C}_1^2(\text{U}_Y(1))]$$

$$I=\text{Isospin} \quad Y=\text{Hypercharge}=2(C-I_2)$$

From R.Bijker

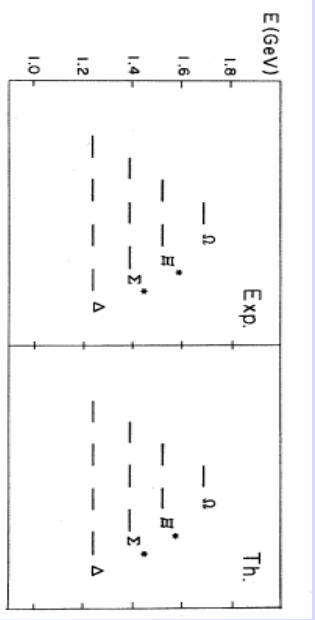


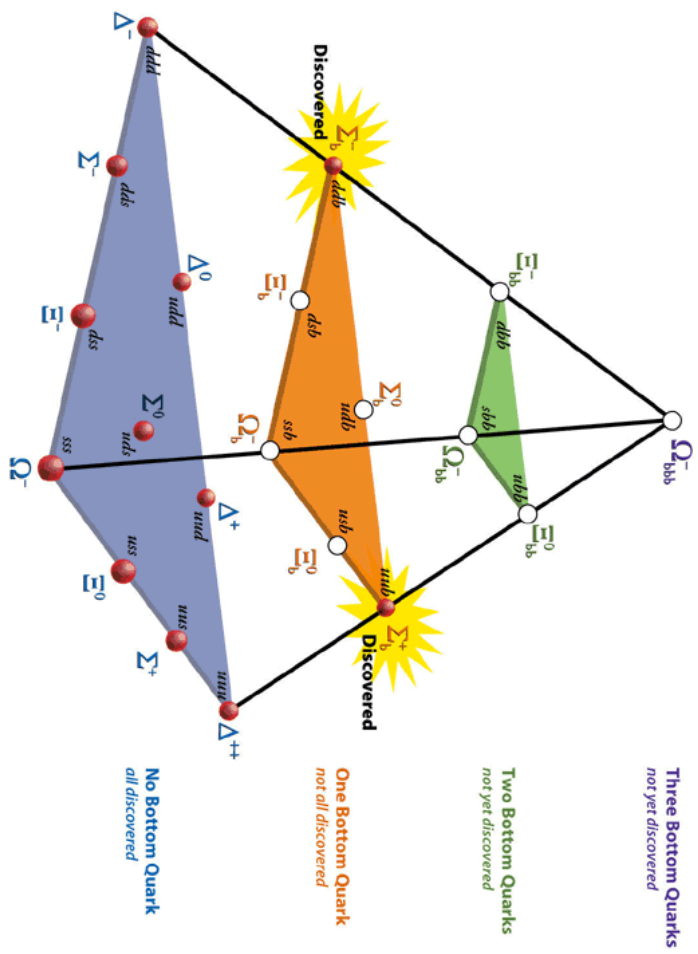
FIG. 3. Level diagram of the baryon decuplet. The energy levels are computed using the mass formula (3.6).

$$M(I,I_3,Y)=M_0+aY+b[I(I+1)-Y^2/4] \quad (3.6)$$

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# Algebraic models... give a scheme!

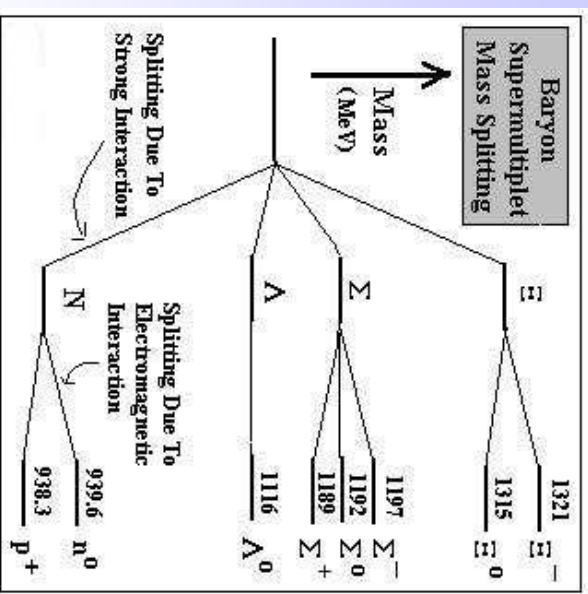
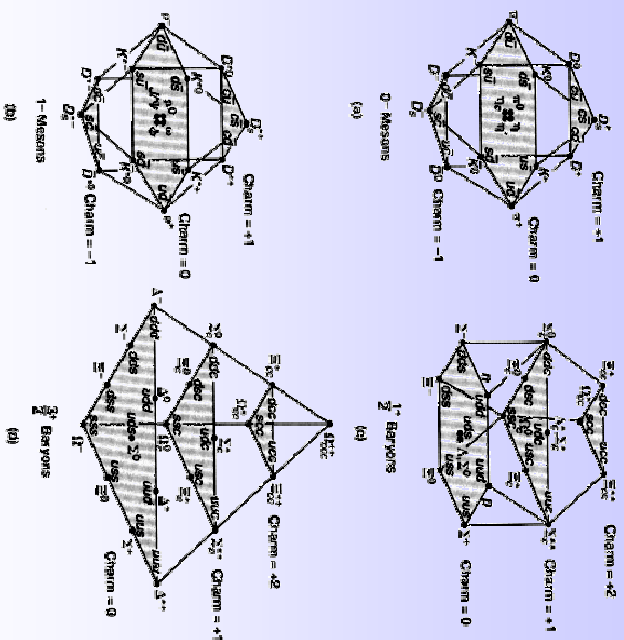
Baryons with Up, Down, Strange and Bottom Quarks and Highest Spin ( $J = \frac{3}{2}$ )



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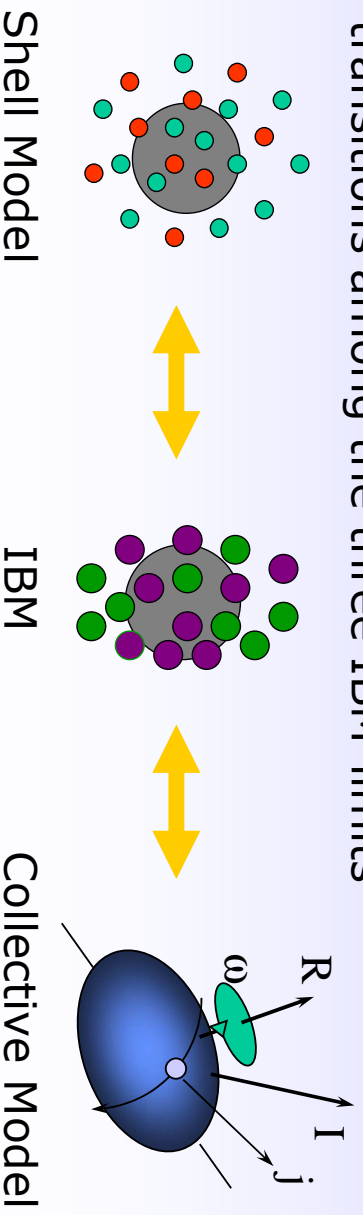
# $SU(3)$ .... explains observation!



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## 2- IBM, $u(6)$ and Dynamical symmetries in nuclei

- Introduction to the nucleus as a manybody system
- Interacting Boson Model (IBM)  $\leftrightarrow$   $u(6)$
- Dynamical Symmetries in the IBM:  $u(5)$   $su(3)$   $so(6)$
- Casten's triangle
- Examples of energy spectra and electromagnetic transitions among the three IBM-limits



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## Interacting Boson Model (IBM o IBA)

$u(6) \rightarrow SGA$  for the atomic nucleus  
It has three dynamical symmetries



Akito Arima and Francesco Iachello

Nucleus: many-body system made up of fermions ( $1 \sim 250$ ) with spin  $1/2$  of 2 species, protons and neutrons.

Fermions have a tendency to couple into composites bosons (analogous to Cooper's pairs in semiconductors) with total angular momentum 0 or 2, called **s** and **d** bosons respectively.

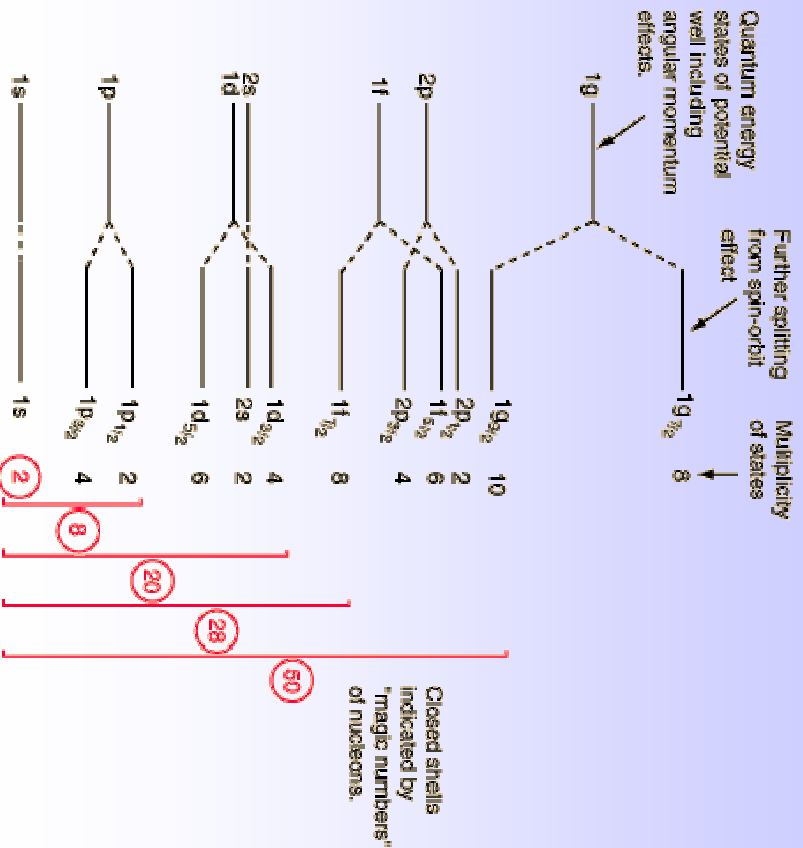
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From R. Bijker

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## Motivations



**Valence nucleons** have a tendency to form couples with  $L=0,2$ . Indeed the fundamental state of even-even nuclei is always  $0^+$ , while the first excited state is almost always a  $2^+$  state.

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## Motivations – 2

Collective states in a nucleus with  $2N$  valence nucleons are approximated by a  $N$ -boson state:

$$N = n_s + n_d$$

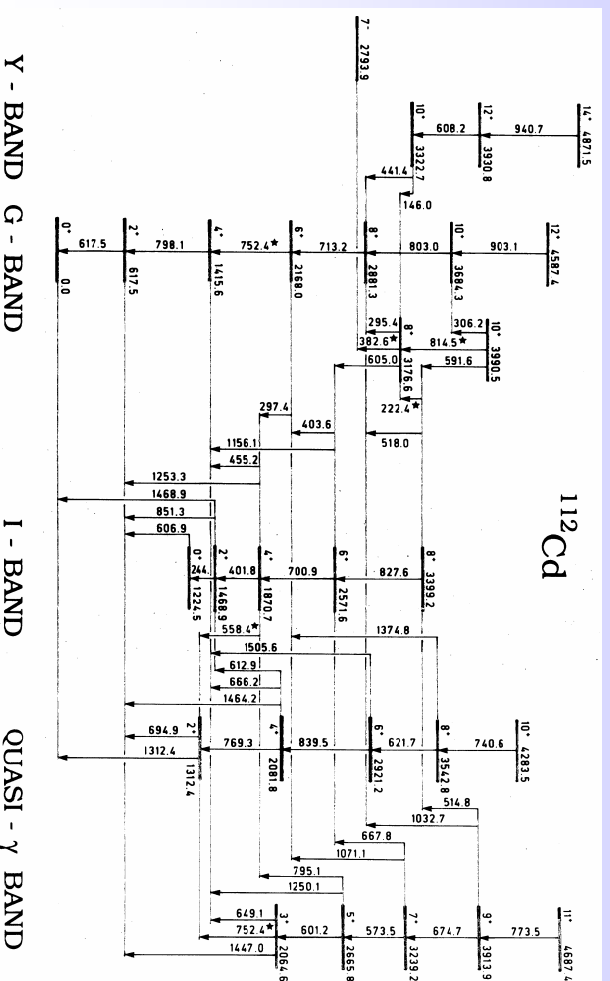
where  $n_s$  and  $n_d$  are not necessarily conserved, but the total boson number  **$N$  is conserved**.

If the shell is up to half-filled one considers **couples of particles**, if it is filled more than a half one considers **couples of holes**.

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## Motivations – 3

There is the need to find a conceptual scheme that allows to set some “order” into the complexity of nuclear spectroscopy → symmetries!



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## Fermions' coupling

Fermions ( $\mathbf{s}=1/2$ ) in a valence shell ( $\mathbf{j}=\ell+\mathbf{s}$ ) are coupled, usually, in such a way that  $\mathbf{J}_{\text{tot}}=0,2$

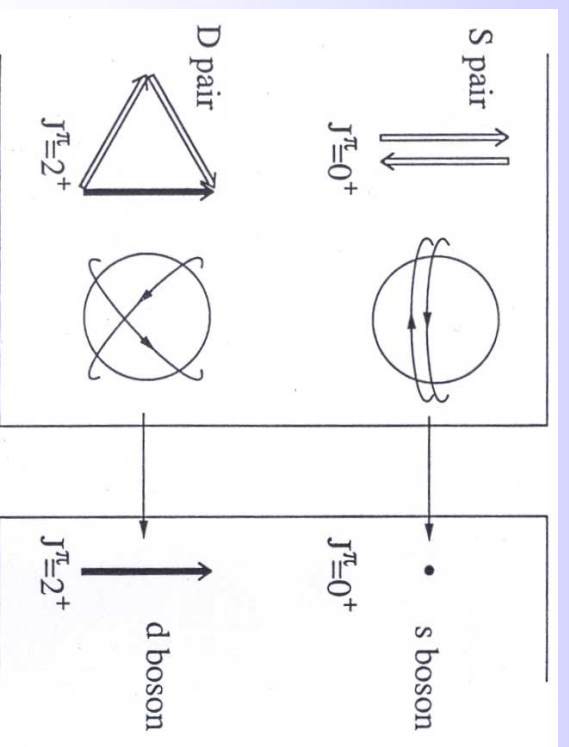


Figure 4.6. Correspondence between nucleon pairs, S and D, and bosons, s and d.

From T.Otsuka

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## Definitions:

“Elementary” bosons (IBM building blocks):

$b_\alpha$ ,  $b_\alpha^\dagger$  with  $\alpha=1,\dots,6$

$b_1 = s$

$b_{2,\dots,6} = d_\mu$

$$\begin{aligned} [b_\alpha, b_{\alpha'}^\dagger] &= \delta_{\alpha\alpha'}, \\ [b_\alpha, b_{\alpha'}] &= [b_\alpha^\dagger, b_{\alpha'}^\dagger] = 0 \end{aligned}$$

One constructs the  $u(6)$  algebra by taking **bilinear** operators:

$$G_{\alpha\beta}^B = b_\alpha^\dagger b_\beta \quad \alpha, \beta = 1, \dots, n \quad \text{order } n^2$$

Such that they close into  $u(6)$ :  $[G_r, G_s] = \sum_t c_t G_t$

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## Vibrational limit $u(6) \supset u(5)$

Elements :	Dimension:	Algebra:
$[s^\dagger \times d]^{(2)}_\mu \rightarrow 5$	36	$u(6)$
$[d^\dagger \times s]^{(2)}_\mu \rightarrow 5$		
$[s^\dagger \times s]^{(0)}_0 \rightarrow 1$		
$[d^\dagger \times d]^{(0)}_0 \rightarrow 1$	25	$u(5)$
$[d^\dagger \times d]^{(2)}_\mu \rightarrow 5$		
$[d^\dagger \times d]^{(4)}_\mu \rightarrow 9$		
$[d^\dagger \times d]^{(3)}_\mu \rightarrow 7$	10	$so(5)$
$[d^\dagger \times d]^{(1)}_\mu \rightarrow 3$	3	$so(3)$

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Rotational limit  $u(6) \supset su(3)$

Elements :                      Dimension:                      Algebra:

$[s^\dagger \times d]^{(2)}_\mu \rightarrow 5$	36	$u(6)$
$[d^\dagger \times s]^{(2)}_\mu \rightarrow 5$		
$[s^\dagger \times s]^{(0)}_0 \rightarrow 1$		
$[d^\dagger \times d]^{(0)}_0 \rightarrow 1$		
$[d^\dagger \times d]^{(3)}_\mu \rightarrow 7$		
$[d^\dagger \times d]^{(4)}_\mu \rightarrow 9$		

$[s^\dagger \times d]^{(2)}_\mu + [d^\dagger \times s]^{(2)}_\mu$		
$- \sqrt{7/2} [d^\dagger \times d]^{(2)}_\mu \rightarrow 5$	8	$su(3)$
$[d^\dagger \times d]^{(1)}_\mu \rightarrow 3$	3	$so(3)$

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$\gamma$ -unstable limit  $u(6) \supset so(6)$

Elements :                      Dimension:                      Algebra:

$[s^\dagger \times d]^{(2)}_\mu \rightarrow 5$	36	$u(6)$
$[s^\dagger \times s]^{(0)}_0 \rightarrow 1$		
$[d^\dagger \times d]^{(0)}_0 \rightarrow 1$		
$[d^\dagger \times d]^{(2)}_\mu \rightarrow 5$		
$[d^\dagger \times d]^{(4)}_\mu \rightarrow 9$		

$[s^\dagger \times d]^{(2)}_\mu + [d^\dagger \times s]^{(2)}_\mu \rightarrow 5$	15	$so(6)$
$[d^\dagger \times d]^{(3)}_\mu \rightarrow 7$	10	$so(5)$
$[d^\dagger \times d]^{(1)}_\mu \rightarrow 3$	3	$so(3)$

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## Subalgebra chains:

$$SU(6) \supset \begin{cases} SU(5) \supset SO(5) \supset SO(3) & \text{(vibrational)} \\ SU(3) \supset SO(3) & \text{(rotational)} \\ SO(6) \supset SO(5) \supset SO(3) & (\gamma\text{-unstable}) \end{cases}$$

With the operators defined in the model one can therefore build **only 3 subalgebra chains**, that are called **limits**. They correspond to dynamical symmetries: for each one it is possible to write a hamiltonian operator (actually not only one) that is **analtically solvable**.

To say it all there's a fourth chain, called  $\overline{su(3)}$ , that is isomorphical to  $su(3)$  (*automorphism*).

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## Chain I: $u(6) \supset u(5) \supset so(5) \supset so(3)$

$$\hat{H}_{u(5)} = \eta_1 \hat{C}_1[u(5)] + \kappa_1 \hat{C}_2[u(5)] + \kappa_4 \hat{C}_2[so(5)] + \kappa_5 \hat{C}_2[so(3)]$$

$$n_d \quad n_d(n_d+4) \quad v(v+3) \quad L(L+1)$$

$$E(n_d, v, L) = \varepsilon n_d + \kappa_1 n_d(n_d + 4) + \kappa_4 v(v + 3) + \kappa_5 L(L + 1)$$

$$\left| \begin{matrix} u(6) \supset u(5) \supset so(5) \supset so(3) \\ N \quad n_d \quad v \quad L \end{matrix} \right\rangle$$

Basis states are labelled by quantum numbers coming from the Casimir operators (+missing label problem)

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Chain II:  $u(6) \supset su(3) \supset so(3)$

$$\hat{H}_{su(3)} = \kappa_2 \hat{C}_2[su(3)] + \kappa_5 \hat{C}_2[so(3)]$$

$$(\lambda, \mu) \quad L(L+1)$$

$$E(\lambda, \mu, L) = \kappa_2 (\lambda^2 + \mu^2 + 3\lambda + 3\mu + \lambda\mu) + \kappa_5 L(L+1)$$

$$\left| \begin{array}{c} u(6) \supset su(3) \supset so(3) \\ N \quad (\lambda, \mu) \quad L \end{array} \right\rangle$$

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Chain III:  $u(6) \supset so(6) \supset so(5) \supset so(3)$

$$\hat{H}_{so(6)} = \kappa_3 \hat{C}_2[so(6)] + \kappa_4 \hat{C}_2[so(5)] + \kappa_5 \hat{C}_2[so(3)]$$

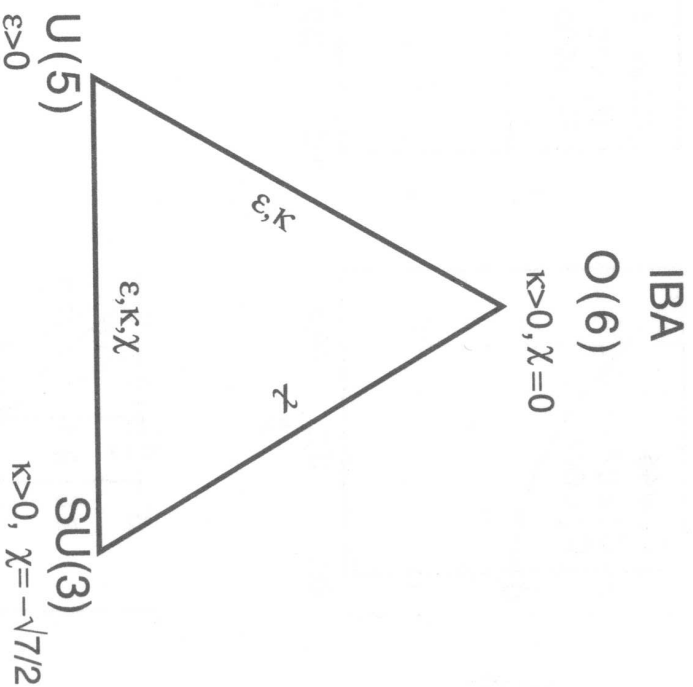
$$\sigma(\sigma+4) \quad \nu(\nu+3) \quad L(L+1)$$

$$E(\sigma, \nu, L) = \kappa_3 [N(N+4) - \sigma(\sigma+4)] + \kappa_4 \nu(\nu+3) + \kappa_5 L(L+1)$$

$$\left| \begin{array}{c} u(6) \supset so(6) \supset so(5) \supset so(3) \\ N \quad \sigma \quad \nu \quad L \end{array} \right\rangle$$

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## Casten's triangle



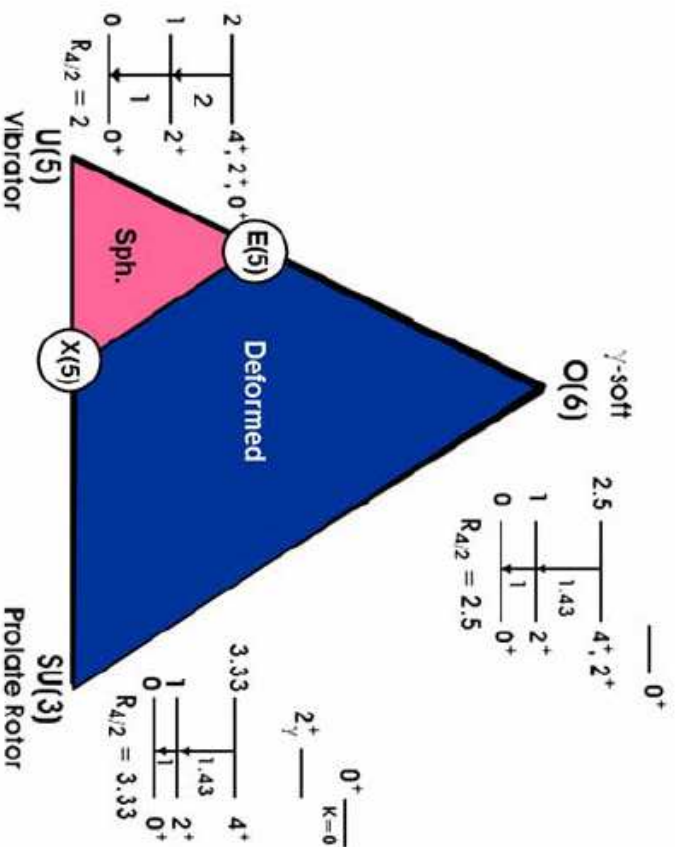
$$H = \epsilon n_d - \kappa Q(\chi) \cdot Q(\chi)$$

The parameters refer to a particular hamiltonian, called "Q dot Q"

The triangle is like a map that shows the whole nuclear phenomenology that is hidden behind quadrupole deformations

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## Casten's triangle



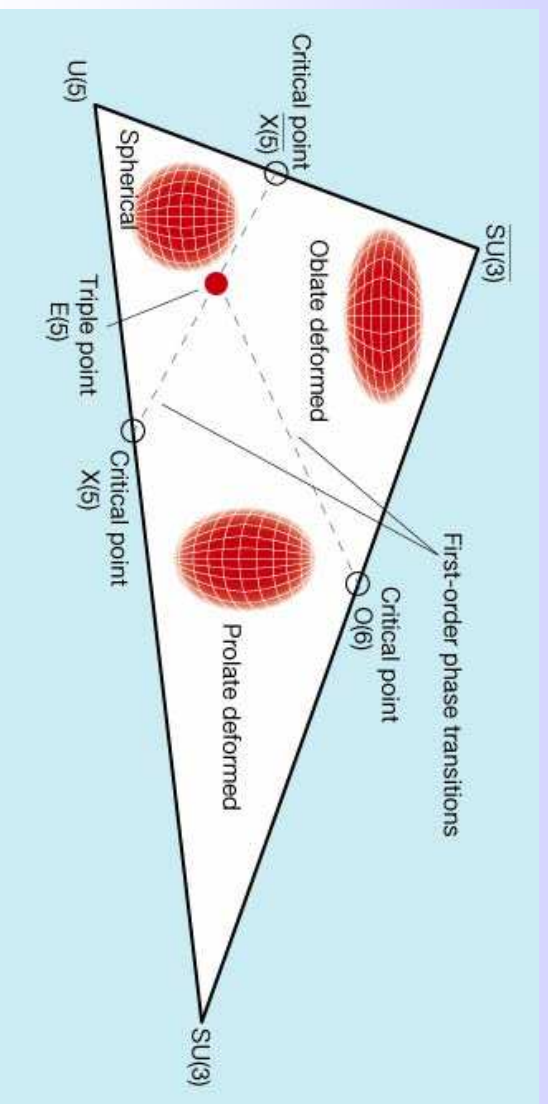
Rick Casten,  
YALE

It has a rich phenomenology !

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## Extended Casten's triangle

It has the chain IV that corresponds to an **oblate** axial rotor instead of a **prolate** one.



There are various phase transitions !

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Spherical, Oblate and Prolate...what the hell?

Spherical

Oblate

Prolate



Orange

Mandarin

Lemon



Soccer

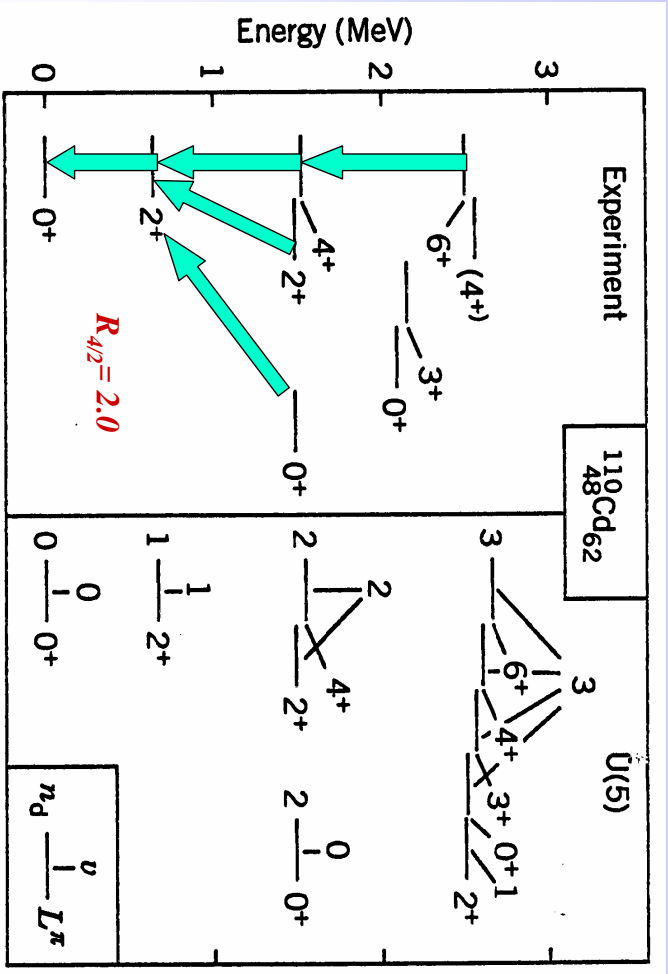
Discus throwing

Rugby

S  
P  
O  
R  
T

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# U(5) – Spherical



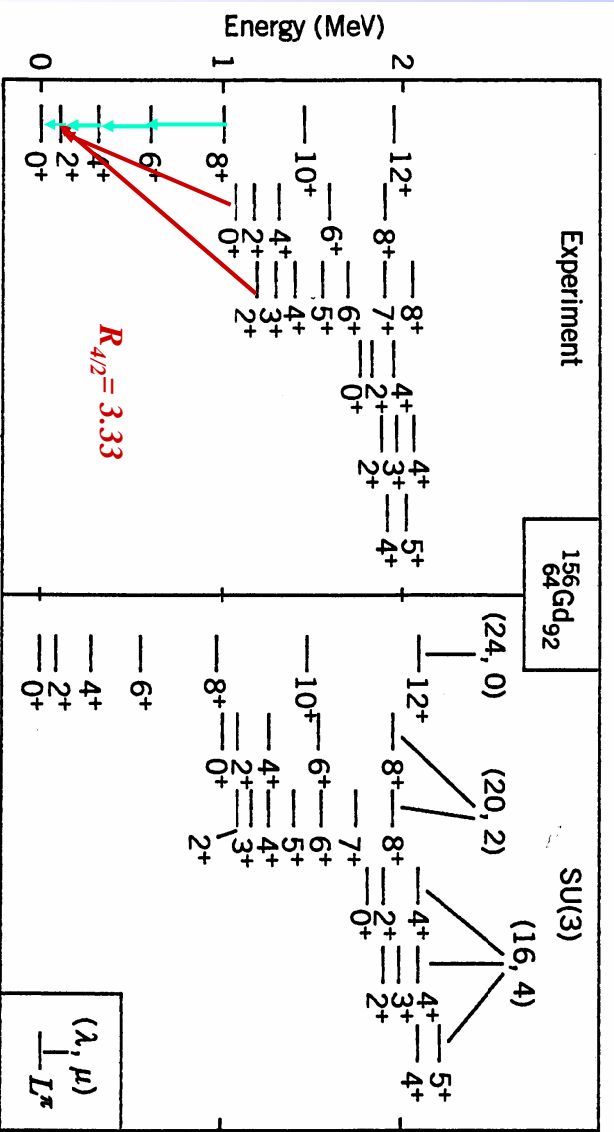
$$B(E2; 2^+ \rightarrow 0^+) = e_B^2 N$$

$$\frac{B(E2; 4^+ \rightarrow 2^+)}{B(E2; 2^+ \rightarrow 0^+)} = e_B^2 2(N-1)$$

Slide by D. Warner

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# SU(3) – Rotational (axial)



$$B(E2; 2^+ \rightarrow 0^+) = e_B^2 \frac{N(2N+3)}{5}$$

$$\frac{B(E2; 4^+ \rightarrow 2^+)}{B(E2; 2^+ \rightarrow 0^+)} = \frac{10}{7} \left[ \frac{(2N-2)(2N+5)}{2N(2N+3)} \right]$$

Slide by D. Warner

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## Crucial Observables -2

- 2-neutron Separation Energy  $E(N+1)-E(N)$
- Position of excited bands, other energy ratios  $R_{0+/2+}$

- Isomer shift  $\delta\langle r^2 \rangle = \langle r^2 \rangle_{2_1^+} - \langle r^2 \rangle_{0_1^+}$

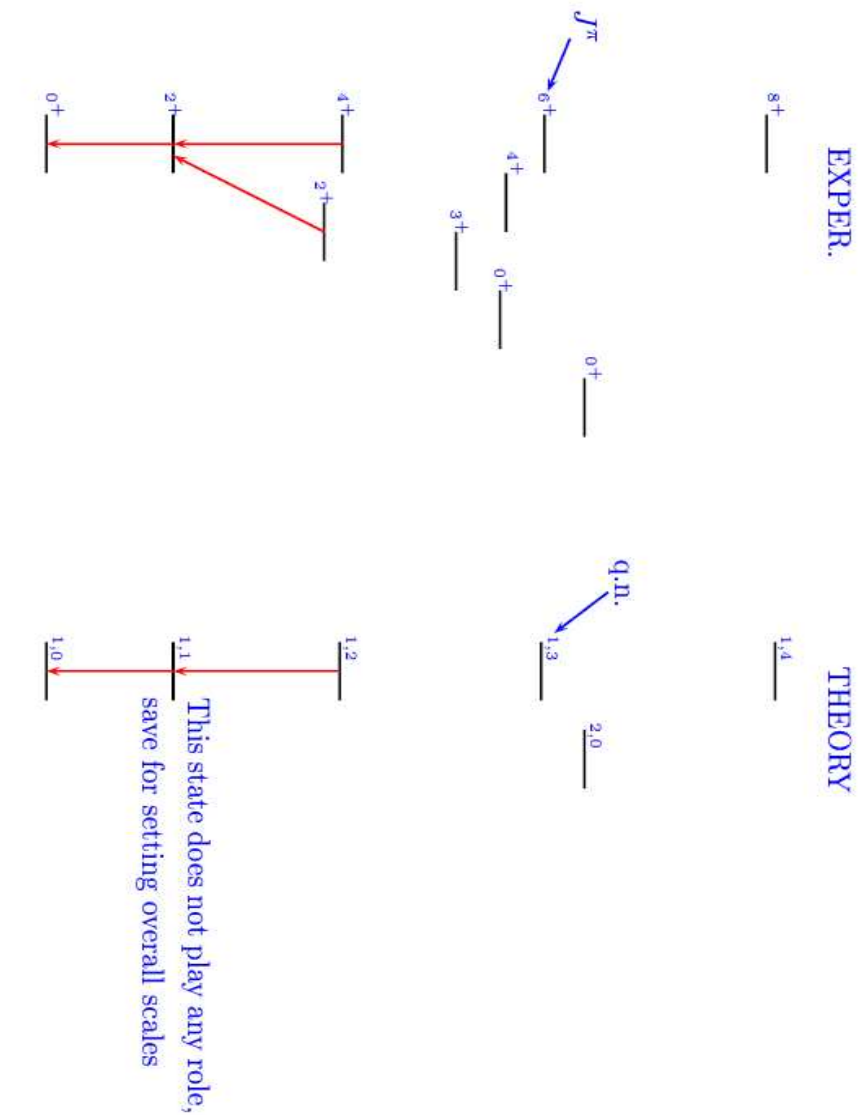
- Intensity of transfer reactions

- Electromagnetic Transitions  $B(E2; 2_1^+ \rightarrow 0_1^+)$

- B(E2) ratios  $B(E0; 0_2^+ \rightarrow 0_1^+)$

E.M. transitions satisfy certain **selection rules** that are dictated by the symmetry of the system and characterize it!

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The idea is to compare data with “sets of rules”, which depend on the symmetry

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## Electromagnetic Transitions and Selection Rules

Operators in general (and the EM transition operator in particular) can also be expressed in terms of the elements of the algebra and one can calculate matrix elements analitically.

For exemple in the electric quadrupole case E2:

$$Q^{(2)}_m = [s^\dagger \times d + d^\dagger \times s + \chi d^\dagger \times d]^{(2)}_m$$

The calculation of **reduced transition probability**:

$$B(E2; L \rightarrow L') = (2L'+1)/(2L+1) |\langle L' || T(E2) || L \rangle|^2$$

implies **selection rules**. For instance in the U(5) limit:

$$B(E2; [N], n_d+1, v=n_d+1, L=2n_d+2 \rightarrow [N], n_d, v'=n_d, L'=2n_d) = k(2N-L')(L'+2)$$

Therefore some transitions are forbidden!

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## 3-IBM2, other extensions and IBFM

- Nucleus as a system of two kinds of particles
- Interacting Boson Model 2 (IBM-2)  $\leftrightarrow$   $u(6) \oplus u(6)$
- Dynamical Symmetries in IBM-2 and F-spin
- Other extensions (brief overview)
- Examples: energy spectra, scissor mode
- Interacting Boson-Fermion Model (IBFM)
- Superalgebra, dynamical supersymmetry, supermultiplets

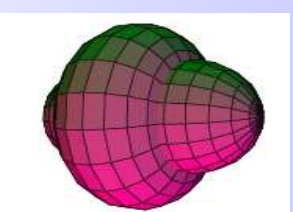
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## IBM-2: protons and neutrons

In the IBM-2 we consider protons and neutrons as **different species**, defining creation and annihilation operators for each one of them:

$$s_{\pi} \ s_{\nu} \ d_{\pi} \ d_{\nu} \quad s_{\pi}^{\dagger} \ s_{\nu}^{\dagger} \ d_{\pi}^{\dagger} \ d_{\nu}^{\dagger}$$



The 72 generators close into  $U_{\pi}(6) \otimes U_{\nu}(6)$  and can be regrouped in 2 subsets:

$$[b_{\pi\ell m}^{\dagger} \times b_{\pi\ell' m'}^{(\lambda)}]_{\mu}^{(\lambda)} \rightarrow 36 \text{ proton operators}$$

$$[b_{\nu\ell m}^{\dagger} \times b_{\nu\ell' m'}^{(\lambda)}]_{\mu}^{(\lambda)} \rightarrow 36 \text{ neutron operators}$$

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## Algebra chains

All algebras are bosonic. Besides trivial chains that involve only separately chains of each of the two algebras  $U_{\pi}(6)$  and  $U_{\nu}(6)$ , we have

$$U_{\pi}(6) \otimes U_{\nu}(6) \supset U_{\pi\nu}(6) \supset 3 \text{ chains} \supset SO_{\pi\nu}(3)$$

The generators  $U_{\pi\nu}(6)$  are obtained by summing the corresponding generators of  $U_{\pi}(6)$  and  $U_{\nu}(6)$ .

Example:  $L_{\pi,i}$  generate  $SO_{\pi}(3)$

$L_{\nu,i}$  generate  $SO_{\nu}(3)$

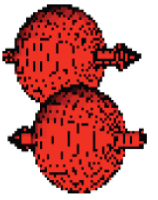
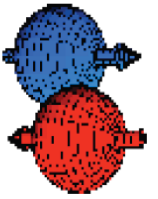
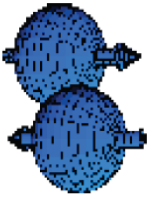
$L_{\pi,i} + L_{\nu,i}$  generate  $SO_{\pi\nu}(3)$

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## F-Spin

By analogy with the isospin formalism we can think to the proton and neutron bosons as two different “charge states” of the same boson with  $F=1/2$  and (by convention)

$$M_F = \begin{matrix} -1/2 & \text{neutron} \\ +1/2 & \text{proton} \end{matrix}$$

$$-^1S_0 \text{ isovector or spin singlet } (S=0, T=1): \hat{S}_+ = \sum_{m>0} \hat{a}_{m\downarrow}^+ \hat{a}_{m\uparrow}^+$$




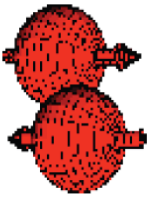
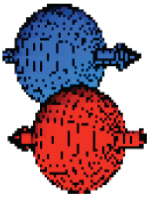
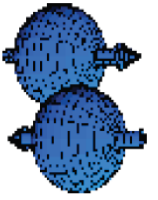
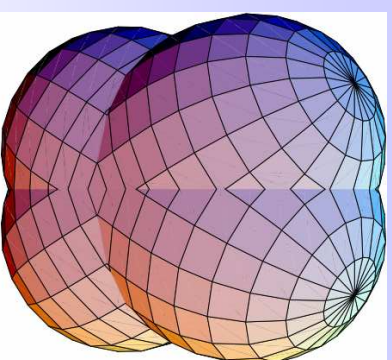
$$-^3S_1 \text{ isoscalar or spin triplet } (S=1, T=0): \hat{P}_+ = \sum_{m>0} \hat{a}_{m\uparrow}^+ \hat{a}_{m\uparrow}^+$$




Figure taken from P. van Isacker

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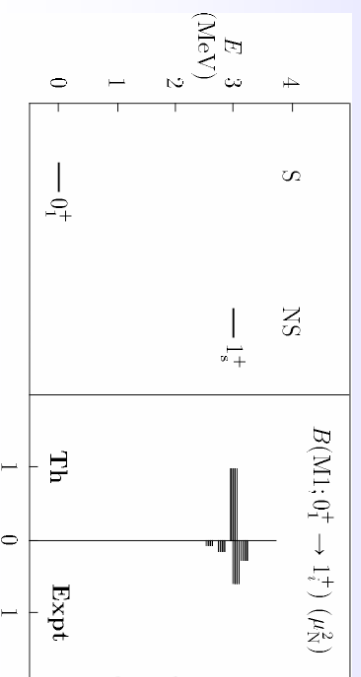
## What does it add to phenomenology?

Predicts scissors states  $L^\pi=1^+$  (**scissors mode**): in the classical limit they are described as collective oscillations of the angle between the symmetry axes of the proton and neutron ellipsoids. They are isovector modes.



They are probed measuring the magnetic  $M1$  transitions

$$L^\pi \rightarrow (L+1)^\pi$$



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$L_{\pi,i}^{-}L_{\nu,i}$  : is the op. that promotes M1 transitions

On top of  $L^{\pi}=1^{+}$  states there are bands with  $L^{\pi}=1^{+}, 2^{+}, 3^{+}, \dots$

## Mixed Symmetry States

These states are called **mixed symmetry states** because they correspond to IRREPS that are not totally symmetric nor totally antisymmetric under the exchange of protons and neutrons:

$$[N_{\pi}] \otimes [N_{\nu}] \supset [N_{\pi}+N_{\nu}, 0, 0, \dots] \oplus [N_{\pi}+N_{\nu}, -1, 1, 0, \dots] \oplus \oplus [N_{\pi}+N_{\nu}, -2, 2, 0, \dots] \oplus \dots$$

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## IBM-2 hamiltonian and proton-neutron interaction

The generic form of a bosonic hamiltonian in the IBM2 is:

$$H = E_0 + \epsilon_{\pi} n_{\pi} + \epsilon_{\nu} n_{\nu} + \kappa Q_{\pi}^{(2)} Q_{\nu}^{(2)} + V_{\pi\pi} + V_{\pi\nu} + V_{\nu\nu} + M_{\pi\nu}$$

$\epsilon_{\pi}$ ,  $\epsilon_{\nu}$  : are linked to the energy of the proton and neutron bosons. Other terms describe the mutual interaction between the two components, in which the proton-neutron **quadrupole interaction** dominates.

In particular then, **the Majorana operator**  $M_{\pi\nu}$  makes possible the transition (shift) between different symmetry states.

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## Other extensions: IBM- $n$ with $n=1,2,3,4$

There are several other extensions and generalizations, some coming from physical considerations, other from mathematical generalizations, but the algebra becomes more and more complicated. We won't deal with them.

- IBM-1: single type of pair.
- IBM-2:  $T=1$  nn ( $M_T=-1$ ) and pp ( $M_T=+1$ ) pairs.
- IBM-3: full isospin  $T=1$  triplet of nn ( $M_T=-1$ ), np ( $M_T=0$ ) and pp ( $M_T=+1$ ) pairs.
- IBM-4: full isospin  $T=1$  triplet and  $T=0$  np pair (with  $S=1$ ).

Taken from P.van Isacker

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## IBFM: Interacting Boson Fermion Model

The IBM and all its extensions treat collective excitations of systems with an even number of particles (even nuclei) in terms of a set of interacting bosons.

It is interesting to extend (Iachello & Scholten, 1979) the model to odd nuclei:

- all particles, except one, are coupled into  $s$  and  $d$  bosons
- the remaining unpaired particle is treated explicitly as a **fermion** that is moving in some given s.p. orbital



F. Iachello

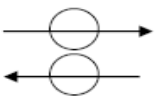


Olaf Scholten

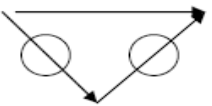
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## IBM and IBFM

In odd systems, besides coupled fermions with  $L=0,2$  there is also an unpaired fermion that interacts with the rest of valence bosons:  $V_{BB}$   $V_{BF}$



S ( $J=0$ ) pairing  $\Rightarrow$  s-boson



D ( $J=2$ ) pairing  $\Rightarrow$  d-boson



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## IBFM, Superalgebra and Supersymmetry

Set of bosonic and fermionic operators that satisfy certain rules  $\rightarrow$  **Superalgebra** o **graded Lie algebra**

Set:  $X_a, Y_b \in \mathcal{G}^*$ ,  $a = 1, \dots, r$ ,  $b = 1, \dots, s$ ,

$X \rightarrow$  bosonic elements  $Y \rightarrow$  fermionic elements

1) Commutation/anticommutation relations:  
(c,d,f are “graded” structure constants)

$$\begin{aligned} [X_a, X_b] &= c_{ab}^c X_c, & c_{ab}^c &= -c_{ba}^c \\ [X_a, Y_b] &= d_{ab}^c Y_c, & d_{ab}^c &= -d_{ba}^c \\ \{Y_a, Y_b\} &= f_{ab}^c X_c, & f_{ab}^c &= +f_{ba}^c \end{aligned}$$

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2) Jacobi Super-identity:

$$\begin{aligned} & [[X_a, X_b], X_c] + [[X_b, X_c], X_a] + [[X_c, X_a], X_b] = 0 \\ & [[X_a, [X_b, Y_c]] + [X_b, [Y_c, X_a]] + [Y_c, [X_a, X_b]] = 0 \\ & [X_a, \{Y_b, Y_c\}] + \{Y_b, [Y_c, X_a]\} - \{Y_c, [X_a, Y_b]\} = 0 \\ & [Y_a, \{Y_b, Y_c\}] + [Y_b, \{Y_c, Y_a\}] + [Y_c, \{Y_a, Y_b\}] = 0 \end{aligned}$$

U(n/m) can be realized with creation and annihilation bilinear operators

$G_{\alpha\beta}^B = b_{\alpha}^{\dagger} b_{\beta}$	$\alpha, \beta = 1, \dots, n$	order $n^2$
$G_{ij}^F = a_i^{\dagger} a_j$	$i, j = 1, \dots, m$	order $m^2$
$F_{i\alpha}^{\dagger} = a_i^{\dagger} b_{\alpha}$	$i = 1, \dots, m, \quad \alpha = 1, \dots, n$	order $mn$
$F_{\alpha i} = b_{\alpha}^{\dagger} a_i$	$\alpha = 1, \dots, n, \quad i = 1, \dots, m$	order $mn$

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Classification of superalgebras

Their classification is known (1975-77)

TABLE 2. Kac classification of classical graded Lie algebras.

Name	Label	Kac label
Special Unitary	$su(n/m)$	$A(n-1, m-1)$
Orthosymplectic	$osp(n/2m), n = \text{odd}$	$B((n-1)/2, m)$
Orthosymplectic	$osp(n/2m), n = \text{even}$	$D(n/2, m)$
Others	$C[n], n \geq 2$	$C[n]$
	$A[n], n \geq 2$	$A[n]$
	$P[n], n \geq 2$	$P[n]$
	$F[4]$	$F[4]$
	$G[3]$	$G[3]$
Non-semisimple	$D[1, 2; \alpha]$	$D[1, 2; \alpha]$
Unitary	$u(n/m)$	



Victor Kac,  
matematico  
russo

L. Fortunato



## Extension

One can extend all the concepts that we have seen to superalgebras:

- there are **super-representations**
- there are **Casimir super-operators** that commute with all the elements of the superalgebra
- there are **spectrum generating superalgebras**



$$H = f(X_a, Y_b) \ , \quad X_a, Y_b \in \mathcal{G}^*$$

- there are **dynamical supersymmetries**

$$H = f(C_i^*)$$

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## IBFM Operators

Bosons :  $b_\alpha^\dagger (\alpha = 1, \dots, 6) \equiv s^\dagger, d_\mu^\dagger (\mu = 0, \pm 1, \pm 2),$

Fermions :  $a_i^\dagger (i = 1, \dots, \Omega) \equiv a_{jm_j}^\dagger (m_j = \pm j, \pm(j-1), \dots, \pm \frac{1}{2})$

$\Omega$  is the dimension of the fermionic space

$$\Omega = \sum_j (2j+1)$$

$$H_B = E_0 + \sum_{\alpha\beta} \varepsilon_{\alpha\beta} b_\alpha^\dagger b_\beta + \sum_{\alpha\alpha'\beta\beta'} u_{\alpha\alpha'\beta\beta'} b_\alpha^\dagger b_{\alpha'}^\dagger b_\beta b_{\beta'}$$

$$H_F = \mathcal{E}_0 + \sum_{ik} \eta_{ik} a_i^\dagger a_k + \sum_{i\ddagger k k'} v_{i\ddagger k k'} a_i^\dagger a_{\ddagger}^\dagger a_k a_{k'}$$

$$V_{BF} = \sum_{\alpha\beta i k} w_{\alpha\beta i k} b_\alpha^\dagger b_\beta^\dagger a_i^\dagger a_k$$

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## IBM + fermion with $j=3/2$

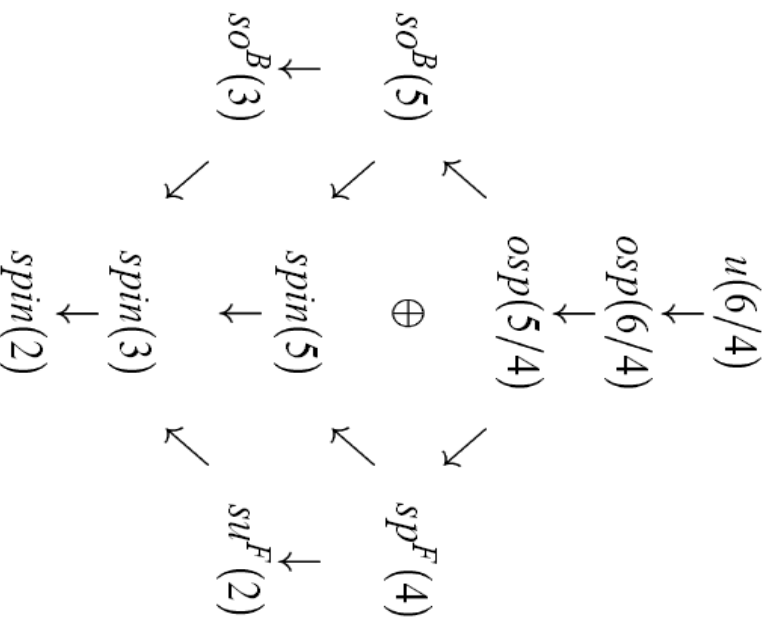
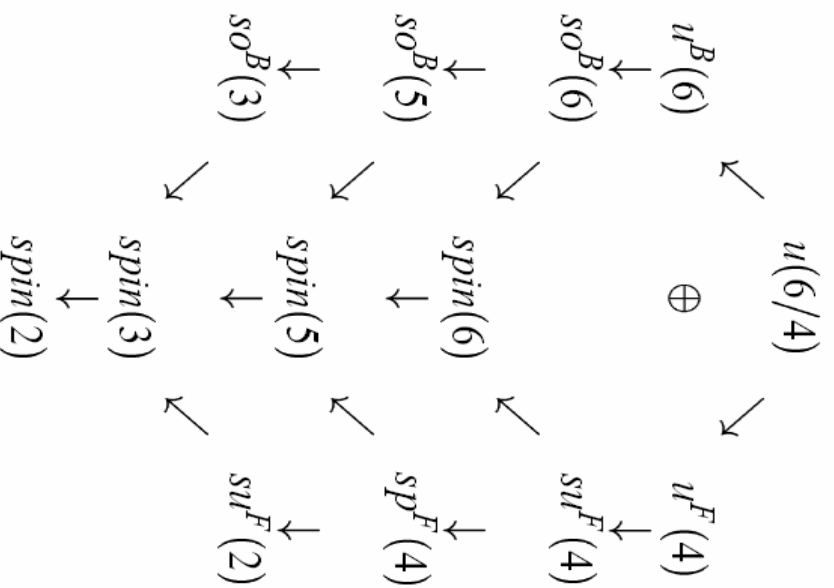
Consider the case of the IBM + a fermion in a  $j=3/2$  orbit (similar to Bayman-Silverberger supersymm.):  
the Lie superalgebra is  $U(6/4)$

The Racah form of the algebra  $u(6/4)$  is

$$\begin{aligned} G_{\kappa}^{(k)}(\ell, \ell') &= \left[ b_{\ell}^{\dagger} \times \tilde{b}_{\ell'} \right]_{\kappa}^{(k)}, \\ A_{\kappa}^{(k)}(j, j) &= \left[ a_j^{\dagger} \times \tilde{a}_j \right]_{\kappa}^{(k)}, \\ F_{\kappa}^{(k)}(j, \ell) &= \left[ a_j^{\dagger} \times \tilde{b}_{\ell} \right]_{\kappa}^{(k)}, \\ B_{\kappa}^{(k)}(\ell, j) &= \left[ b_{\ell}^{\dagger} \times \tilde{a}_j \right]_{\kappa}^{(k)}. \end{aligned}$$

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## Examples of "Lattice" or superalgebras network



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## Examples of “Lattice” or superalgebras network – 2

$U^B(6) \supset \dots \supset SO^B(3)$  for Bosons

$U^F(4) \supset \dots \supset SU^F(3)$  for Fermions

The two chains are combined into a single chain

$U^B(6) \otimes U^F(4) \supset SO^B(6) \otimes SU^F(4) \supset$

$Spin(6) \supset Spin(5) \supset Spin(3)$

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## Classification of states

The chains gives a way to classify (label) states

$$\left| \begin{array}{ccccccc}
 u(6/4) & \supset & u^B(6) \oplus u^F(4) & \supset & so^B(6) \oplus su^F(4) & & \\
 \downarrow \mathcal{N} & & \downarrow N_B, N_F & & \downarrow \Sigma & & \\
 & \supset & spin(6) & \supset & spin(5) & \supset & spin(3) & \supset & spin(2) \\
 & & \downarrow (\sigma_1, \sigma_2, \sigma_2) & & \downarrow (\tau_1, \tau_2) & & \downarrow v_\Delta, J & & \downarrow M_J
 \end{array} \right\rangle$$

$\mathcal{N}, N_B, N_F, \Sigma, \sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, v_\Delta, J, M_J$  Are the needed q.n.

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## Example of Hamiltonian and analytic spectrum

$$H = e_0 + e_6 C_1(u6/4) + e_7 C_2(u6/4) + e_1 C_1(u^B 6) + e_2 C_2(u^B 6) + e_3 C_1(u^F 4) + e_4 C_2(u^F 4) + e_5 C_1(u^B 6) C_1(u^F 4) + \eta C_2(o^B 6) + \eta' C_2(spin6) + \beta C_2(spin5) + \gamma C_2(spin3) ,$$

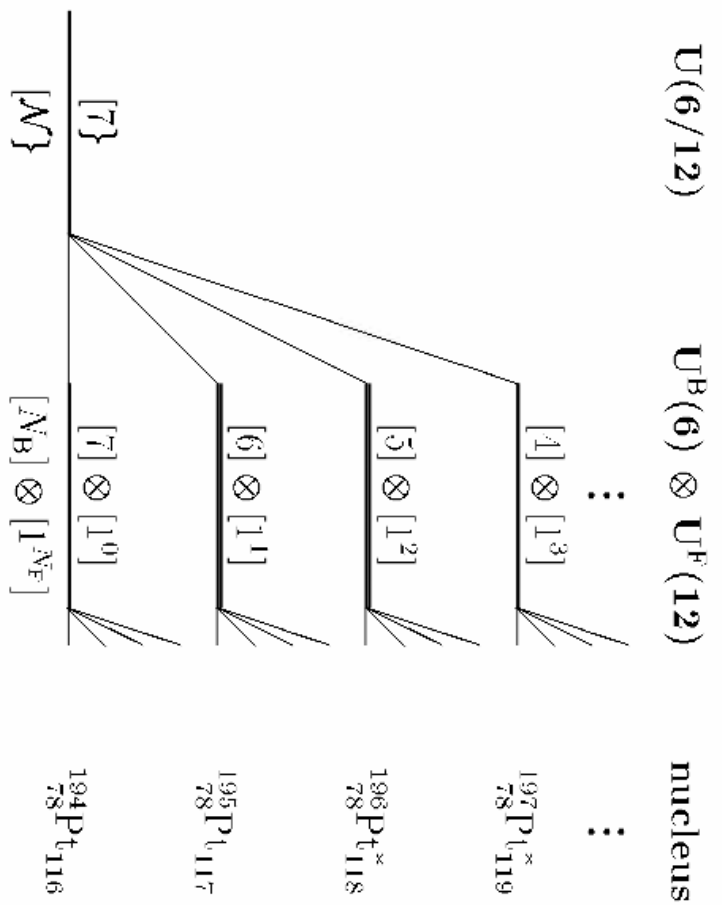
with eigenvalues

$$\begin{aligned} E &\equiv E(\mathcal{N}, N_B, N_F, \Sigma, (\sigma_1, \sigma_2, \sigma_2), (\tau_1, \tau_2), \nu_\Delta, J, M_J) \\ &= e_0 + e_6 \mathcal{N} + e_7 \mathcal{N}(\mathcal{N} + 1) + e_1 N_B + e_2 N_B(N_B + 5) + e_3 N_F \\ &\quad + e_4 N_F(5 - N_F) + e_5 N_B N_F + \eta \Sigma(\Sigma + 4) \\ &\quad + \eta' [\sigma_1(\sigma_1 + 4) + \sigma_2(\sigma_2 + 2) + \sigma_3^2] \\ &\quad + \beta [\tau_1(\tau_1 + 3) + \tau_2(\tau_2 + 1)] + \gamma J(J + 1) . \end{aligned}$$

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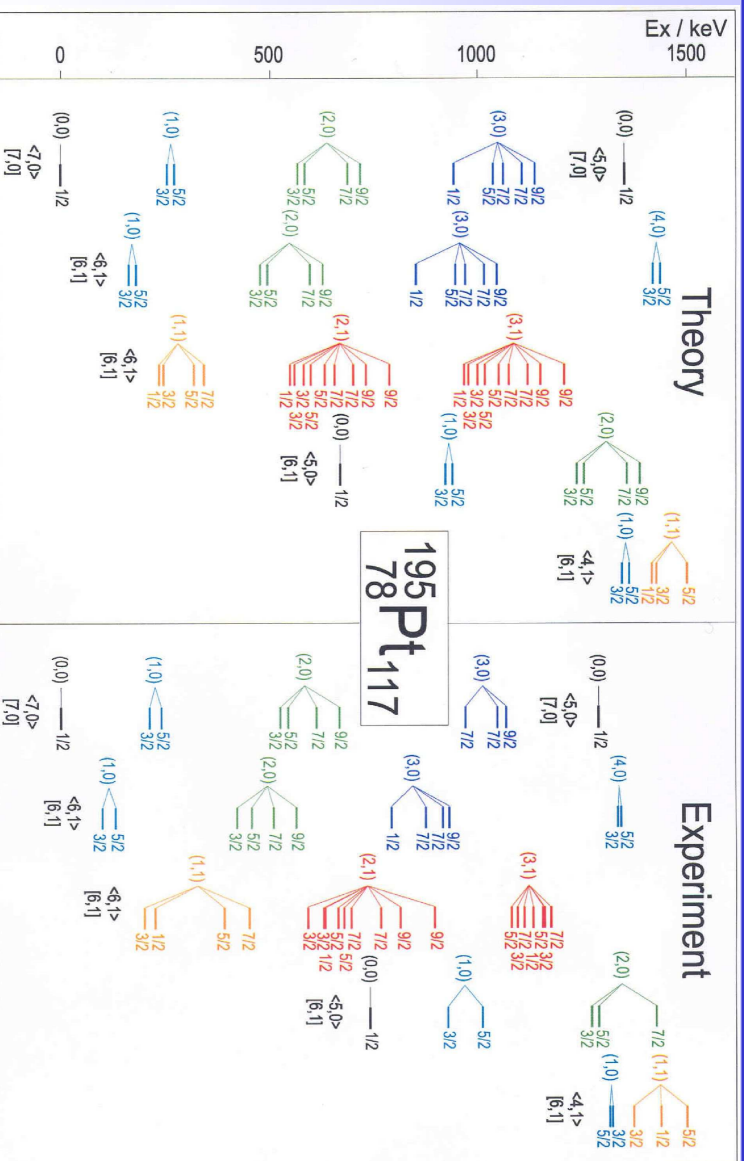
## Supermultiplets $U(6/12)$

Given a certain REP of the supergroup we associate a set of subgroup REP's that correspond to different nuclei !!



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## Comparison between theory and experiment

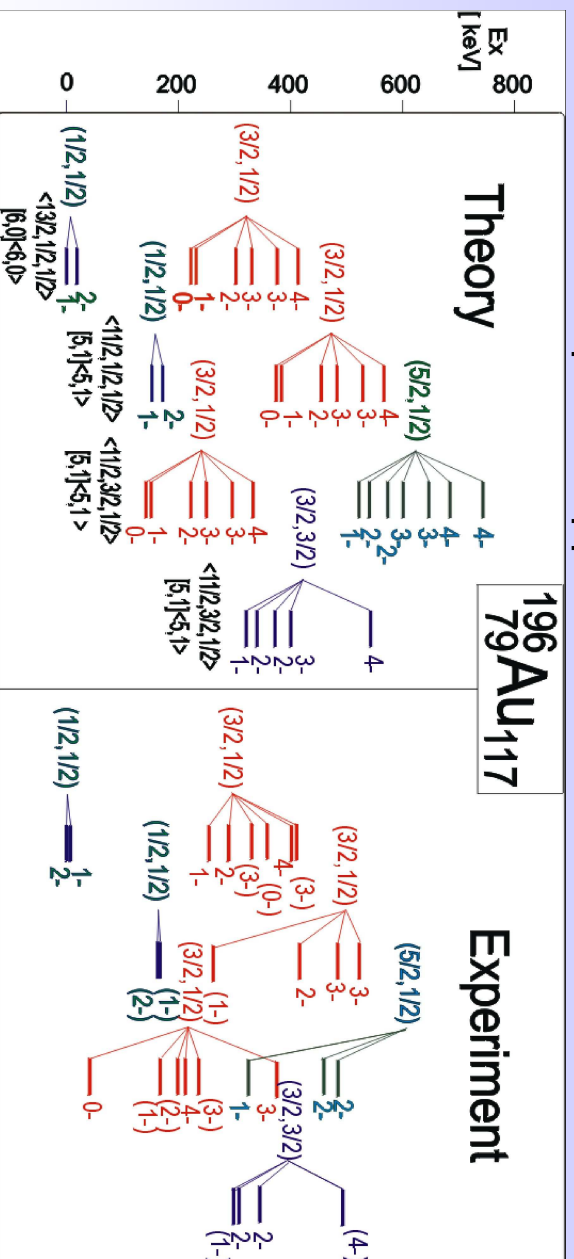


Qualitatively excellent, quantitatively good

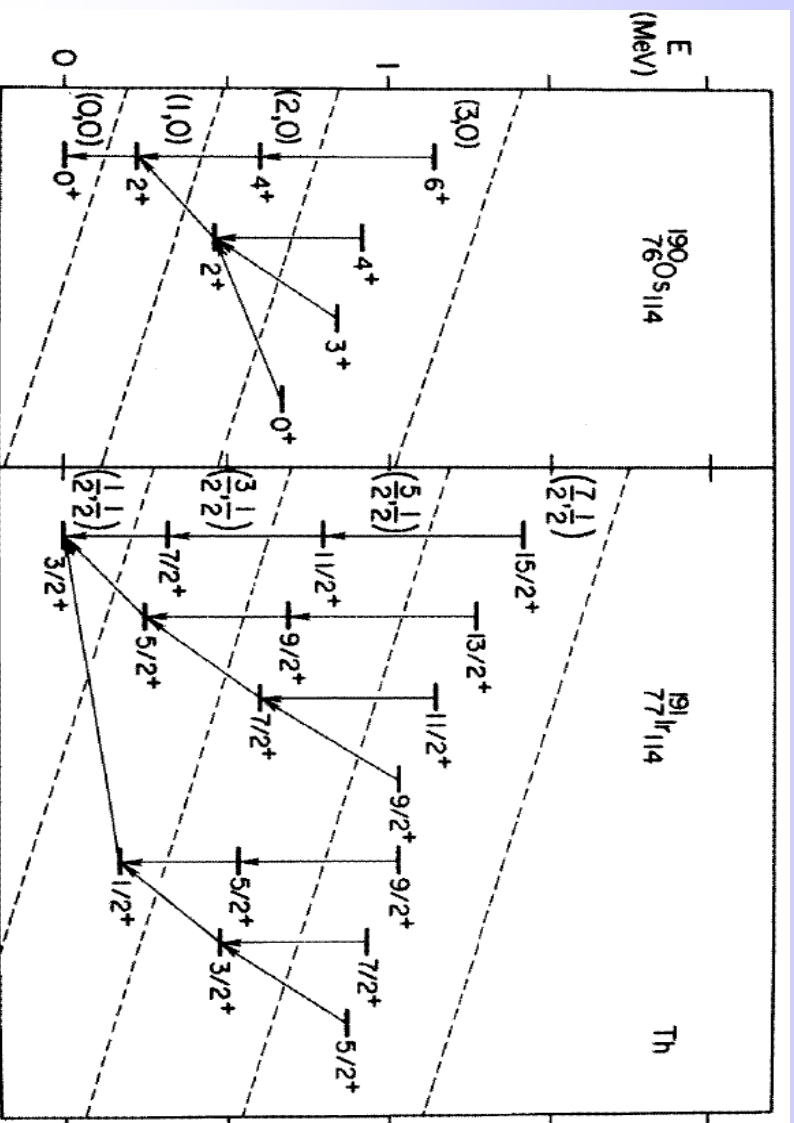
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Other...

Another example of application of IBFM: 196-Au



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## Dynamical proton-neutron Supersymmetry

Even-even		odd-even
$N_\nu + 1, N_\pi + 1$		$N_\nu, N_\pi + 1, j_\nu$
$^{194}\text{Pt}$	$\Leftrightarrow$	$^{194}\text{Pt}$
$\Updownarrow$		$\Updownarrow$
$^{195}\text{Au}$	$\Leftrightarrow$	$^{196}\text{Au}$
$N_\nu + 1, N_\pi, j_\pi$		$N_\nu, N_\pi, j_\nu, j_\pi$
even-odd		odd-odd

$$U(6/12)_\nu \otimes U(6/4)_\pi$$

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## One nucleon transfer reactions

The **transfer reactions** of a single nucleon (neutron or proton) furnish a crucial test for supersymmetry. The transfer operator is:

$$T_1 = -\sigma[s_\pi \times a^\dagger_{\pi, 3/2}]^{(3/2)}_m + \theta[d_\pi \times a^\dagger_{\pi, 3/2}]^{(3/2)}_{m \text{ gs}}$$
$$T_2 = \theta[s_\pi \times a^\dagger_{\pi, 3/2}]^{(3/2)}_m + \sigma[d_\pi \times a^\dagger_{\pi, 3/2}]^{(3/2)}_{m \text{ gs, exc}}$$

It is a tensor under transformations induced by Spin(6), hence obeys selection rules and gives analytic expressions for the transfer intensities.

Example :  $^{194}\text{Pt} \rightarrow ^{195}\text{Au}$

Theoretical and experimental values (in some reduced unities)

$$R_{\text{exc}}=0 \qquad \text{Exp}=0.019$$

$$R_{\text{gs}} = 1.12 \qquad \text{Exp}=1.175$$

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## Message to be taken with you

- Some **simple** models are “naturally” written in terms of creation and annihilation operators.
- To them we can always associate an algebra that brings with itself a **dynamical symmetry**.
- By knowing how to deal mathematically with the algebra one can get analytic solutions that can be compared with experimental data (not only: you can get also **new, unexpected solutions!**)
- The algebra naturally entails **quantum numbers** (=classification), **selection rules** (= explain some weird observations).
- **The algebra gives a conceptual frame** and might give hints on **new physics!**

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