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Lecture II

An excercise in ν oscillation probability calculations

Goal:

Derive the approximate expression of $P(\nu_e \to \nu_\mu)$ in matter for 3ν oscillations, at 2nd order in the small parameters $\sin\theta_{13}$ and δm^2

(used in many papers on ν-factory and superbeam ν physics)

Donini, Meloni, Rigolin ....
Yokomakura, Kimura, Nakamura ...

Conventions & Notations

Fields: $\nu_{\alpha L} = \sum_i U_{\alpha i}\, \nu_{iL}$ $\quad (i = 1,2,3)$ $\quad (\alpha = e, \mu, \tau)$

flavor mixing mass

1-particle states: $|\nu_\alpha\rangle = \sum_i U^*_{\alpha i}\, |\nu_i\rangle$

(need $\bar{\psi}$, not ψ, to create a particle from vacuum $|0\rangle$)

Components: $|\nu\rangle = \sum_\alpha \nu^\alpha |\nu_\alpha\rangle = \sum_i \nu^i |\nu_i\rangle$

numbers

$$\nu^\alpha = \sum_i U_{\alpha i}\, \nu^i \qquad !$$

Unfortunately, there is much confusion in the literature about use of fields, states, components and thus about U/U^* conventions.

In the following, we'll use vector and matrix components in flavor basis.

PDG convention :

$$U = O_{23}\,\Gamma_\delta\,O_{13}\,\Gamma_\delta^+\,O_{12}$$

$$\Gamma_\delta = \mathrm{diag}(1,1,e^{i\delta})$$

$\delta = \cancel{CP}$ phase

$$O_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}$$

$c_{ij} = \cos\theta_{ij}$

$s_{ij} = \sin\theta_{ij}$

$$O_{13} = \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix}$$

$$O_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Some authors put only Γ_δ or Γ_δ^+, not both.
Better to put Γ_δ <u>and</u> Γ_δ^+, so that $\det U = 1$ instead of $\det(U) = e^{\pm i\delta}$.

Ranges : $\theta_{ij} \in [0, \frac{\pi}{2}]$

$\delta \in [0, 2\pi]$

Explicitly:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

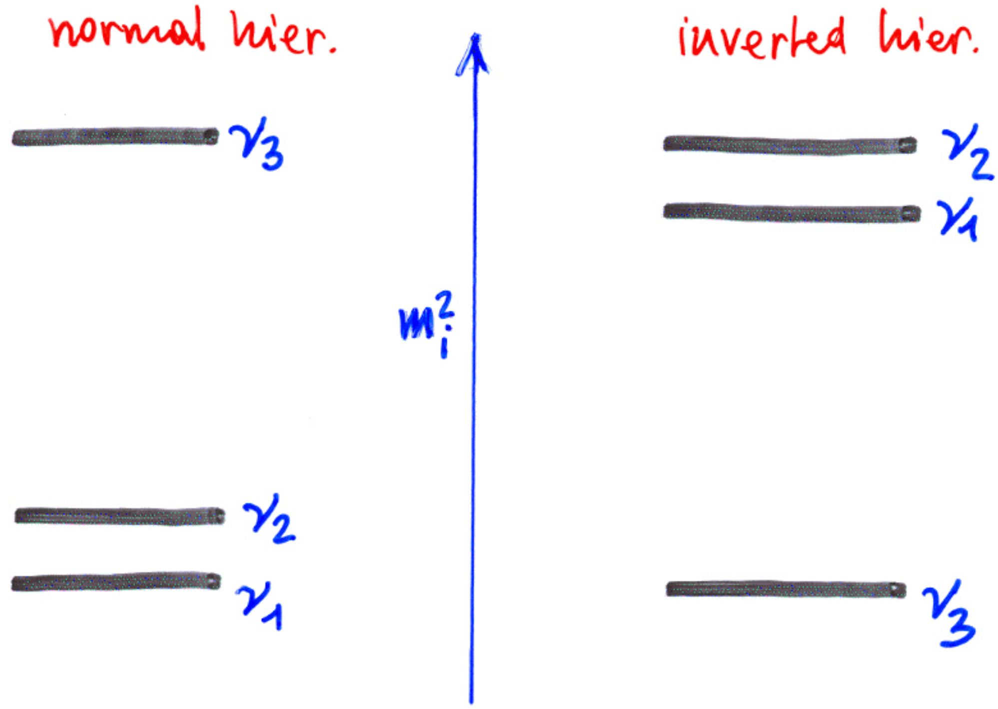
$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

For antineutrinos, $U \to U^*$

i.e., $\delta \to -\delta$

In the following, we shall refer to neutrinos

Neutrino mass convention



- Almost all authors agree in labeling the "lone" state as ν_3, and the "doublet" as (ν_1, ν_2)

- There is also agreement in labeling ν_2 as the heaviest of (ν_1, ν_2) in both hierarchies, so that

  $$\delta m^2 = m_2^2 - m_1^2 > 0$$

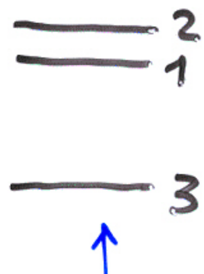
- Then the two hierarchies are distinguished by the physical sign of $m_3^2 - m_{1,2}^2$:

$$\text{sign}\,(m_3^2 - m_{1,2}^2) = \begin{cases} +1 & \text{(normal)} \\ -1 & \text{(inverted)} \end{cases}$$

The problem is how to define the 2nd (larger) squared mass difference.

Some authors choose $m_3^2 - m_1^2$, others $m_3^2 - m_2^2$.
Both choices have a drawback: To swap hierarchy, it is not sufficient to change $m_3^2 - m_1^2$ into $m_1^2 - m_3^2$ (i.e., just swap sign):

$m_3^2 - m_1^2$ is the largest m^2 difference



$-(m_3^2 - m_1^2)$ is the next-to-largest m^2 difference

→ If you use $m_3^2 - m_1^2$ in normal hierarchy, must use $m_2^2 - m_3^2$ in inverse hierarchy (i.e, swap sign & change one label)

This "problem" is almost irrelevant in current phenomenology, but is starting to emerge in prospective studies of physics at future ν-factories

Our solution is to define the "large" m^2 differ. as the average of $m_3^2 - m_1^2$ and $m_3^2 - m_2^2$:

With this convention, changing hierarchy is exactly equivalent to swap $+\Delta m^2 \to -\Delta m^2$ without any re-labeling

Squared mass matrix:

$$\mathcal{M}^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$$

$$\stackrel{\text{def}}{=} \frac{m_1^2 + m_2^2}{2} + \left(-\frac{\delta m^2}{2}, +\frac{\delta m^2}{2}, \pm \Delta m^2\right)$$

$$m_2^2 - m_1^2 = \delta m^2$$

$$m_3^2 - m_1^2 = \pm\Delta m^2 + \frac{\delta m^2}{2}$$

$$m_3^2 - m_2^2 = \pm\Delta m^2 - \frac{\delta m^2}{2}$$

$+$: normal
$-$: inverted

In the following: $+\Delta m^2$ used.

Hamiltonian matrix components in flavor basis $|\nu_\alpha\rangle$, in vacuum:

$$H = \frac{1}{2E} \cdot U \mathcal{M}^2 U^+ = \frac{1}{2E} U \begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix} U^+$$

... and in matter (MSW effect)

$$H = \frac{1}{2E} U \mathcal{M}^2 U^+ + V$$

$$V = \mathrm{diag}\left(\sqrt{2}\, G_F N_e(x),\ 0,\ 0\right)$$

N_e = electron density

$\sqrt{2} G_F N_e = V_e - V_{\mu,\tau}$ = interaction energy difference

In the following: N_e = const (approx. valid along Earth crust trajectories).

Auxiliary variable: $A = 2EV = 2\sqrt{2}\, G_F N_e E$

(for $\bar{\nu}$: $V \to -V$; in the following: V)

In constant-density matter, the evolution operator is simple,

$$S(x,0) = e^{-iHx}$$

$\left(\text{insted of } S = \mathcal{T}\left[\int e^{-iH\,dt}\,dt\right] \right)$

If we are able to diagonalize

$$H = \frac{1}{2E} U \mathcal{M}^2 U^+ + V$$

$$= \frac{1}{2E} \tilde{U} \tilde{\mathcal{M}}^2 \tilde{U}^+$$

diagonal

then:

$$S(x,0) = \tilde{U}\, e^{-i\frac{\mathcal{M}^2}{2E}x}\, \tilde{U}^+$$

But: $\dim(H) = 3$

→ cubic secular equation

→ formally simple but messy in practice, if analytical and tractable expressions are needed

So, better to try approximate diagonalization → enormous literature!

Most of the tricks reduce to:

1) Use a suitable basis to simplify the problem

2) Reduce the 3ν evolution to a 2ν evolution when possible

3) Expand in small parameters

We shall use all these tricks to calculate

$$P(\nu_e \to \nu_\mu)$$

Change of basis

Let us recall that:

$$H = U \frac{\mathcal{M}^2}{2E} U^\dagger + V$$

$$\mathcal{M}^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$$

$$U = O_{23}\, \Gamma_\delta\, O_{13}\, \Gamma_\delta^\dagger\, O_{12}$$

$$\Gamma_\delta = \mathrm{diag}(1, 1, e^{i\delta})$$

$$V = \mathrm{diag}(\sqrt{2}\, G_F N_e, 0, 0)$$

in flavor basis.

A basis where calculations are simpler is given by the (complex) rotation:

$$\nu' = (O_{23}\, \Gamma_\delta)^\dagger\, \nu$$

rotation

new "flavor" components

old (flavor) components

In fact, using the properties ...

$$(O_{23}\, \Gamma_\delta)^\dagger\, V\, (O_{23}\, \Gamma_\delta) \equiv V$$

$$\Gamma_\delta^\dagger\, O_{12}\, \mathcal{M}^2\, O_{12}^T\, \Gamma_\delta = O_{12}\, \mathcal{M}^2\, O_{12}^T$$

... one can easily see that the hamiltonian H' in the new "flavor" basis is:

$$H' = (O_{23}\Gamma_\delta)^+ H (O_{23}\Gamma_\delta)$$

$$= O_{13} O_{12} \frac{\mathcal{M}^2}{2E} (O_{13} O_{12})^T + V$$

→ H' does not depend on δ and thus is real symmetric: $H'_{\alpha\beta} = H'_{\beta\alpha}$

→ H' does not depend on θ_{23}

→ $S' = e^{-iH'x}$
- is symmetric ($S'_{\alpha\beta} = S'_{\beta\alpha}$)
- does not depend on δ
- does not depend on θ_{23}

So, we first calculate $S' = e^{-iH'x}$ and then get S by rotating back to the flavor basis:

$$S = (O_{23}\Gamma_\delta) S' (O_{23}\Gamma_\delta)^+$$

$$S = (O_{23}\Gamma_\delta)\, S' (O_{23}\Gamma_\delta)^+$$

$$O_{23}\Gamma_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23}e^{i\delta} \\ 0 & -s_{23} & c_{23}e^{i\delta} \end{pmatrix}$$

Actually we need only $S_{e\mu}$ to calculate $P(\nu_e \to \nu_\mu)$:

$$P(\nu_e \to \nu_\mu) = |S_{e\mu}|^2$$

namely,

$$S_{e\mu} = S'_{e\mu} c_{23} + S'_{e\tau} s_{23} e^{-i\delta}$$

It follows that

$$P_{e\mu} = |S_{e\mu}|^2$$
$$= A_{e\mu} \cos\delta + B_{e\mu} \sin\delta + C_{e\mu}$$

with

$$A_{e\mu} = 2\,\mathrm{Re}\,[S'_{e\mu} S'_{e\tau}]\, c_{23} s_{23}$$
$$B_{e\mu} = -2\,\mathrm{Im}\,[S'_{e\mu} S'_{e\tau}]\, c_{23} s_{23}$$
$$C_{e\mu} = |S'_{e\mu}|^2 c_{23}^2 + |S'_{e\tau}|^2 s_{23}^2$$

→ We need $S'_{e\mu}$ and $S'_{e\tau}$ now!

The trick is now to reduce the evolution from 3ν to 2ν, by making use of an expansion in two phenomenologically small parameters:

s_{13} ($s_{13}^2 \lesssim$ few % from CHOOZ + ...)

$\frac{\delta m^2}{\Delta m^2}$ ($\sim \frac{1}{30}$)

In the following, a term T will be called "first-order" if proportional to s_{13} or δm^2:

$$T \sim O_1 \quad \text{if} \quad T \propto \begin{matrix} s_{13} \\ \text{or } \delta m^2 \end{matrix} \; ;$$

"second order" if proportional to s_{13}^2, $(\delta m^2)^2$, or $s_{13} \cdot \delta m^2$, etc...

We shall show that $S'_{e\mu} \sim O_1$
$S'_{e\tau} \sim O_1$

Therefore $P_{e\mu}$, being a quadratic form in $S'_{e\mu}$ and $S'_{e\tau}$, will be a good approximation at second order:

$$P_{e\mu} \sim O_2$$

Intermezzo: evolution operator in the 2ν subcase.

$$H_{2\nu} = \frac{1}{2E}\left[\begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix}\begin{pmatrix} m_1^2 & 0 \\ 0 & m_2^2 \end{pmatrix}\begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\right]$$

$$A = 2\sqrt{2}\, G_F N_e E$$

In traceless form:

$$H_{2\nu} = \frac{1}{4E}\begin{bmatrix} A - \cos 2\theta\,\Delta & \sin 2\theta\,\Delta \\ \sin 2\theta\,\Delta & -A + \cos 2\theta\,\Delta \end{bmatrix}$$

$$\Delta = m_2^2 - m_1^2$$

Eigenvalues: $\pm \frac{\tilde{\Delta}}{4E}$

with $\tilde{\Delta} = \Delta\sqrt{\left(\cos 2\theta - \frac{A}{\Delta}\right)^2 + \sin^2 2\theta}$

Diagonalizing rotation:

$$H = \begin{pmatrix} \cos\tilde{\theta} & \sin\tilde{\theta} \\ -\sin\tilde{\theta} & \cos\tilde{\theta} \end{pmatrix}\begin{pmatrix} -\frac{\tilde{\Delta}}{4E} & \\ & +\frac{\tilde{\Delta}}{4E} \end{pmatrix}\begin{pmatrix} \cos\tilde{\theta} & -\sin\tilde{\theta} \\ \sin\tilde{\theta} & \cos\tilde{\theta} \end{pmatrix}$$

with $\rightarrow$

$$\sin 2\tilde{\theta} = \frac{\sin 2\theta}{\sqrt{\left(\cos 2\theta - \frac{A}{\Delta}\right)^2 + \sin^2 2\theta}}$$

$$\cos 2\tilde{\theta} = \frac{\cos 2\theta - \frac{A}{\Delta}}{\sqrt{\left(\cos 2\theta - \frac{A}{\Delta}\right)^2 + \sin^2 2\theta}}$$

Note that: $\tilde{\Delta} \sin 2\tilde{\theta} = \Delta \sin 2\theta$

$\tilde{\Delta}$ = "effective" m^2 difference in matter
$\tilde{\theta}$ = "effective" mixing angle in matter

Now, $H_{2\nu} = \begin{pmatrix} c_{\tilde{\theta}} & s_{\tilde{\theta}} \\ -s_{\tilde{\theta}} & c_{\tilde{\theta}} \end{pmatrix} \begin{pmatrix} -\frac{\tilde{\Delta}}{4E} & \\ & +\frac{\tilde{\Delta}}{4E} \end{pmatrix} \begin{pmatrix} c_{\tilde{\theta}} & -s_{\tilde{\theta}} \\ s_{\tilde{\theta}} & c_{\tilde{\theta}} \end{pmatrix}$

$$\rightarrow S_{2\nu} = e^{-iH_{2\nu}x}$$

$$= \begin{pmatrix} c_{\tilde{\theta}} & s_{\tilde{\theta}} \\ -s_{\tilde{\theta}} & c_{\tilde{\theta}} \end{pmatrix} e^{-i\begin{pmatrix} -\frac{\tilde{\Delta}}{4E} & \\ & +\frac{\tilde{\Delta}}{4E} \end{pmatrix}x} \begin{pmatrix} c_{\tilde{\theta}} & -s_{\tilde{\theta}} \\ s_{\tilde{\theta}} & c_{\tilde{\theta}} \end{pmatrix}$$

(trivial)

$$S_{2\nu} = \cos\left(\frac{\tilde{\Delta}}{4E}x\right) \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - i\sin\left(\frac{\tilde{\Delta}}{4E}x\right) \cdot \begin{bmatrix} -\cos 2\tilde{\theta} & \sin 2\tilde{\theta} \\ \sin 2\tilde{\theta} & \cos 2\tilde{\theta} \end{bmatrix}$$

The flavor transition probability is the square modulus of the off-diagonal amplitude:

$$P_{transition} = |S_{off}|^2$$

where:

$$S_{off} = -i\sin\left(\frac{\tilde{\Delta}x}{4E}\right)\sin 2\tilde{\theta}$$

$$\rightarrow P_{transition} = \sin^2 2\tilde{\theta}\, \sin^2\left(\frac{\tilde{\Delta}x}{4E}\right)$$

just as in vacuum (Pontecorvo), but with $\theta \rightarrow \tilde{\theta}$

$\Delta \rightarrow \tilde{\Delta}$

End of the "Intermezzo"

Let's get back to the 3ν hamiltonian H' in the ν' basis:

$$H' = O_{13} O_{12} \frac{\mathcal{M}^2}{2E} (O_{13} O_{12})^T + V$$

$$\mathcal{M}^2 = \mathrm{diag}\left(-\frac{\delta m^2}{2}, +\frac{\delta m^2}{2}, \Delta m^2\right)$$

$$V = \mathrm{diag}\left(\sqrt{2} G_F N_e, 0, 0\right)$$

The evolution decouples as $3\nu = (2\nu) \oplus (1\nu)$ in two limits:

$s_{13} \to 0 \qquad \to O_{13} \equiv \mathbb{1}$

$\delta m^2 \to 0 \qquad \to O_{12} \mathcal{M}^2 O_{12} = \mathcal{M}^2$

Let us define

$$H^{\ell} = \lim_{s_{13} \to 0} H'$$

$$H^{h} = \lim_{\delta m^2 \to 0} H'$$

$s_{13} \to 0$

$$H^\ell = \lim_{s_{13}\to 0} H' = \frac{1}{2E}\left[O_{12} \begin{pmatrix} -\frac{\delta m^2}{2} & & \\ & +\frac{\delta m^2}{2} & \\ & & \Delta m^2 \end{pmatrix} O_{12}^{\dagger} + \begin{pmatrix} A & & \\ & 0 & \\ & & 0 \end{pmatrix} \right]$$

$$= \frac{A}{4E}\mathbb{1} + \frac{1}{2E}\left[O_{12} \begin{pmatrix} -\frac{\delta m^2}{2} & & \\ & +\frac{\delta m^2}{2} & \\ & & \Delta m^2 \end{pmatrix} O_{12}^{\dagger} + \begin{pmatrix} A/2 & & \\ & -A/2 & \\ & & -A/2 \end{pmatrix} \right]$$

$$= \frac{A}{4E}\mathbb{1} + \frac{1}{4E}\begin{bmatrix} A - \cos 2\theta_{12}\,\delta m^2 & \sin 2\theta_{12}\,\delta m^2 & 0 \\ \sin 2\theta_{12}\,\delta m^2 & \cos 2\theta_{12}\,\delta m^2 - A & 0 \\ 0 & 0 & 2\Delta m^2 - A \end{bmatrix}$$

→ In the "flavor" basis ν', the (e,μ) subsystem evolves separately from the (τ) flavor in H^ℓ

→ $S^\ell_{\tau e} = \lim_{s_{13}\to 0} S'_{\tau e} = 0$

→ $S'_{\tau e}$ vanishes with s_{13}

→ $S'_{\tau e} = \mathcal{O}(s_{13}) = \mathcal{O}_1$ (at least)

On the other hand, for $S^\ell_{\mu e}$ we have:

$$S^\ell_{\mu e} = e^{-i\frac{A}{4E}x} \cdot \left[-i \sin 2\tilde{\theta}_{12} \sin\left(\frac{\delta\tilde{m}^2 x}{4E}\right)\right]$$

with $\sin 2\tilde{\theta}_{12} = \sin 2\theta_{12} \Big/ \sqrt{\left(\cos 2\theta_{12} - \frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}}$

$\delta\tilde{m}^2 = \delta m^2 \sin 2\theta_{12} / \sin 2\tilde{\theta}_{12}$

$\delta m^2 \to 0$

$$H^h = \lim_{\delta m^2 \to 0} H' = \frac{1}{2E}\left(O_{13} \begin{bmatrix} 0 & & \\ & 0 & \\ & & \Delta m^2 \end{bmatrix} O_{13}^T + \begin{bmatrix} A & & \\ & 0 & \\ & & 0 \end{bmatrix} \right)$$

$$= \left(\frac{\Delta m^2}{4E} + \frac{A}{4E}\right) \mathbb{1} + \frac{1}{4E} \begin{bmatrix} A - \cos 2\theta_{13}\Delta m^2 & 0 & \sin 2\theta_{13}\Delta m^2 \\ 0 & \Delta m^2 - A & 0 \\ \sin 2\theta_{13}\Delta m^2 & 0 & \cos 2\theta_{13}\Delta m^2 - A \end{bmatrix}$$

→ In the "flavor" basis ν', the (e,τ) flavors evolve separately from the (μ) on in H^h

→ $S^h_{e\mu} = \lim_{\delta m^2 \to 0} S'_{e\mu} = 0$

→ $S'_{e\mu}$ vanishes with δm^2

→ $S'_{e\mu} = \mathcal{O}(\delta m^2) = \mathcal{O}_1$ at least

On the other hand :

$$S^h_{e\tau} = e^{-i\frac{A}{4E}x}\, e^{-i\frac{\Delta m^2}{4E}x} \left[-i \sin 2\tilde{\theta}_{13} \sin\left(\frac{\Delta \tilde{m}^2 x}{4E}\right)\right]$$

with $$\sin 2\tilde{\theta}_{13} = \frac{\sin 2\theta_{13}}{\sqrt{\left(\cos 2\theta_{13} - \frac{A}{\Delta m^2}\right)^2 + \sin^2 2\theta_{13}}}$$

$$\Delta\tilde{m}^2 = \Delta m^2 \sin 2\theta_{13} / \sin 2\tilde{\theta}_{13}$$

Putting all together:

1) $S'_{e\mu} = \mathcal{O}(\delta m^2)$ and is thus approximated by

$$S^{\ell}_{e\mu} = e^{-i\frac{A}{4E}x}\left[-i\sin 2\hat{\theta}_{12}\,\sin\left(\frac{\delta\hat{m}^2 x}{4E}\right)\right]$$

2) $S'_{e\tau} = \mathcal{O}(s_{13})$ and is thus approximated by

$$S^{h}_{e\tau} = e^{-i\frac{A}{4E}x}\, e^{-i\frac{\Delta m^2}{4E}x}\left[-i\sin 2\hat{\theta}_{13}\,\sin\left(\frac{\Delta\hat{m}^2}{4E}x\right)\right]$$

$$\downarrow$$

$$S'_{e\mu} = \cancel{e^{-i\frac{A}{4E}x}}\left[-i\sin 2\hat{\theta}_{12}\,\sin\left(\frac{\delta\hat{m}^2 x}{4E}\right)\right] + \mathcal{O}_2$$

$$S'_{e\tau} = \cancel{e^{-i\frac{A}{4E}x}}\, e^{-i\frac{\Delta m^2}{4E}x}\left[-i\sin 2\hat{\theta}_{13}\,\sin\left(\frac{\Delta\hat{m}^2}{4E}x\right)\right] + \mathcal{O}_2$$

irrelevant common phase

We have now everything to calculate $P_{e\mu}$ as a quadratic form in $S'_{e\mu}$ and $S'_{e\tau}$. Further tricks involve proper organization of terms and expansion in the small parameter $\frac{\delta m^2}{A} = \frac{\delta m^2}{2\sqrt{2}G_F N_e E}$ ("high energy")

Let us recall that:

$$P_{e\mu} = A_{e\mu} \cos\delta + B_{e\mu} \sin\delta + C_{e\mu}$$

$$A_{e\mu} = 2\,\mathrm{Re}\,[S'^{*}_{\mu e} S'_{\tau e}]\, c_{23} s_{23}$$

$$B_{e\mu} = -2\,\mathrm{Im}\,[S'^{*}_{\mu e} S'_{\tau e}]\, c_{23} s_{23}$$

$$C_{e\mu} = |S'_{\mu e}|^2 c_{23}^2 + |S'_{\tau e}| s_{23}^2$$

$$\begin{pmatrix} S'_{\mu e} = S'_{e\mu} \\ S'_{\tau e} = S'_{e\tau} \end{pmatrix}$$

with $P_{e\mu}$ correctly calculated at θ_2 included, using the previous expressions. Explicitly:

$$A_{e\mu} = \sin 2\tilde{\theta}_{12} \sin 2\tilde{\theta}_{13} \sin 2\theta_{23} \times \sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right) \sin\left(\frac{\delta\tilde{m} x}{4E}\right) \cos\left(\frac{\Delta m^2 x}{4E}\right)$$

$$B_{e\mu} = \sin 2\tilde{\theta}_{12} \sin 2\tilde{\theta}_{13} \sin 2\theta_{23} \cdot \sin\left(\frac{\Delta\tilde{m}^2 x}{4E}\right) \sin\left(\frac{\delta\tilde{m}^2 x}{4E}\right) \sin\left(\frac{\Delta m^2 x}{4E}\right)$$

$$C_{e\mu} = c_{23}^2 \sin^2 2\tilde{\theta}_{12} \sin^2\left(\frac{\delta\tilde{m}^2 x}{4E}\right) + s_{23}^2 \sin^2 2\tilde{\theta}_{12} \sin^2\left(\frac{\Delta\tilde{m}^2 x}{4E}\right)$$

High-energy expansion: $A \gg \delta m^2$

$$\sin 2\tilde{\theta}_{12} = \frac{\sin 2\theta_{12}}{\sqrt{\left(\cos 2\theta_{12} - \frac{A}{\delta m^2}\right)^2 + \sin^2 2\theta_{12}}}$$

$$= \sin 2\theta_{12} \, \frac{\delta m^2}{|A|} + \mathcal{O}_2$$

$$\frac{\delta m^2}{\delta \tilde{m}^2} = \frac{\sin 2\tilde{\theta}_{12}}{\sin 2\theta_{12}} = \frac{\delta m^2}{|A|} + \mathcal{O}_2$$

$$\rightarrow \delta \tilde{m}^2 = |A| + \mathcal{O}_2$$

$$\rightarrow \sin\left(\frac{\delta \tilde{m}^2 L}{4E}\right) \simeq \sin\left(\frac{AL}{4E}\right)$$

$$\sin 2\tilde{\theta}_{13} = \frac{\sin 2\theta_{13}}{\sqrt{\left(\cos 2\theta_{13} - \frac{A}{\Delta m^2}\right)^2 + \sin^2 2\theta_{13}}} = \frac{\sin 2\theta_{13}}{\left|1 - \frac{A}{\Delta m^2}\right|} + \mathcal{O}_2$$

where $\mathcal{O}_2 \propto \begin{cases} (\delta m^2)^2 \\ \delta m^2 \cdot s_{13} \\ s_{13}^2 \end{cases}$

(and we have kept track of absolute values)

The high-energy expansion is basically used to express $\tilde{\theta}_{12}$, $\tilde{\theta}_{13}$, $\Delta\tilde{m}^2$, and $\delta\tilde{m}^2$, in terms of vacuum values:

$$A_{e\mu} \simeq \sin 2\theta_{12} \left(\frac{\delta m^2}{|A|}\right) \sin 2\theta_{13} \left|\frac{\Delta m^2}{\Delta m^2 - A}\right| \sin 2\theta_{23}$$

$$\cdot \sin\left(\frac{|A|x}{4E}\right) \sin\left(\Delta m^2 \left|\frac{\Delta m^2 - A}{\Delta m^2}\right| \frac{x}{4E}\right) \cos\left(\frac{\Delta m^2 x}{4E}\right)$$

$$B_{e\mu} \simeq \sin 2\theta_{12} \left(\frac{\delta m^2}{|A|}\right) \sin 2\theta_{13} \left|\frac{\Delta m^2}{\Delta m^2 - A}\right| \sin 2\theta_{23}$$

$$\cdot \sin\left(\frac{|A|x}{4E}\right) \sin\left(\Delta m^2 \left|\frac{\Delta m^2 - A}{\Delta m^2}\right| \frac{x}{4E}\right) \sin\left(\frac{\Delta m^2 x}{4E}\right)$$

$$C_{e\mu} \simeq c_{23}^2 \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A}\right)^2 \sin^2\left(\frac{Ax}{4E}\right)$$

$$+ s_{23}^2 \sin^2 2\theta_{13} \left(\frac{\Delta m^2}{\Delta m^2 - A}\right)^2 \sin^2\left(\left|\frac{\Delta m^2 - A}{4E}\right| L\right)$$

Note : by changing the sign of $(\Delta m^2 - A)$, $B_{e\mu}$, $A_{e\mu}$ and $C_{e\mu}$ do not change;

by changing the sign of Δm^2, $A_{e\mu}$ changes sign (while $B_{e\mu}$ and $C_{e\mu}$ do not)

$\rightarrow$ can eliminate $|\,,|$ properly

$$A_{e\mu} \cong \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \left(\frac{\delta m^2}{A}\right)\left(\frac{\Delta m^2}{A-\Delta m^2}\right)$$

$$\cdot \sin\left(\frac{Ax}{4E}\right) \sin\left(\frac{A-\Delta m^2}{4E}x\right) \cos\left(\frac{\Delta m^2 x}{4E}\right)$$

$$B_{e\mu} \cong \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \left(\frac{\delta m^2}{A}\right)\left(\frac{\Delta m^2}{A-\Delta m^2}\right)$$

$$\cdot \sin\left(\frac{Ax}{4E}\right) \sin\left(\frac{A-\Delta m^2}{4E}x\right) \sin\left(\frac{\Delta m^2 x}{4E}\right)$$

$$C_{e\mu} \cong c_{23}^2 \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A}\right)^2 \sin^2\left(\frac{Ax}{4E}\right)$$

$$+ s_{23}^2 \sin^2 2\theta_{13} \left(\frac{\Delta m^2}{\Delta m^2 - A}\right)^2 \sin^2\left(\frac{\Delta m^2 - A}{4E}x\right)$$

Terms in $P_{e\mu} = A_{e\mu} \cos\delta + B_{e\mu} \sin\delta + C_{e\mu}$ can finally be organized as:

$$P(\nu_e \to \nu_\mu) = X \sin^2 2\theta_{13}$$
$$+ Y \sin 2\theta_{13} \cdot \cos\left(\delta - \frac{\Delta m^2 x}{4E}\right)$$
$$+ Z$$

with $X = \sin^2\theta_{23} \left(\frac{\Delta m^2}{A - \Delta m^2}\right)^2 \sin^2\left(\frac{A - \Delta m^2}{4E} L\right)$

$L = x$

$$Y = \sin 2\theta_{12} \sin 2\theta_{23} \left(\frac{\delta m^2}{A}\right)\left(\frac{\Delta m^2}{A - \Delta m^2}\right) \cdot \sin\left(\frac{AL}{4E}\right) \sin\left(\frac{A - \Delta m^2}{4E} L\right)$$

$$Z = \cos^2\theta_{23} \sin^2 2\theta_{12} \left(\frac{\delta m^2}{A}\right)^2 \sin^2\left(\frac{AL}{4E}\right)$$

Sometimes you may find a further $\cos\theta_{13}$ factor in Y; it is, however, irrelevant at θ^2 included.

E.g. Donini, Meloni, Migliozzi

Another way to write the same formula is:

$$P_{\mu e} = x^2 f^2 + 2xyfg\cos(\delta + \Delta) + y^2 g^2$$

where

$$x = \sin\theta_{23}\,\sin 2\theta_{13}$$

$$y = \frac{\delta m^2}{\Delta m^2}\cos\theta_{23}\,\sin 2\theta_{12}$$

$$f = \sin\left(\frac{\Delta m^2 - A}{4E}\cdot L\right)\frac{\Delta m^2}{\Delta m^2 - A}$$

$$g = \sin\left(\frac{AL}{4E}\right)\cdot\frac{\Delta m^2}{A}$$

$$\Delta = \frac{\Delta m^2 L}{4E}$$

(E.g. Barger et al.)

Finally, it should be observed that this formula for $P_{e\mu}$ works better than one might expect.

In particular, it gives the correct vacuum limit for $A=0$ (no MSW effect), despite the fact that $A \to 0$ is forbidden in our expansion! ($A \gg \delta m^2$!)

In other words, for $A \to 0$, one gets (luckily) the expression of $P_{e\mu}$ (vacuum) that is obtained (correctly) by expanding the exact vacuum formula at O_2.
(not shown)

$\rightarrow$

vacuum:

$$P_{e\mu}^{vac} \cong s_{23}^2 \sin^2 2\theta_{13} \sin^2\left(\frac{\Delta m^2 x}{4E}\right)$$ ← "atmospheric" term

$$+ c_{23}^2 \sin^2 2\theta_{12} \left(\frac{\delta m^2 x}{4E}\right)^2$$ ← "solar" term

$$+ \left[\cos\theta_{13}\right] \sin 2\theta_{13} \sin 2\theta_{23} \sin 2\theta_{12} \cdot \cos\left(\frac{\Delta m^2 x}{4E} - \delta\right) \sin\left(\frac{\Delta m^2 x}{4E}\right)\left(\frac{\delta m^2 x}{4E}\right)$$

only in some papers (unnecessary at O_2)

"interference" term

Another way to organize the vacuum expression is:

$$P_{e\mu} = |A + B|^2$$
$$= |A|^2 + |B|^2 + 2\,\mathrm{Re}\,AB^*$$
$$= P_{sol} + P_{atm} + P_{interf}$$

with $A = c_{23} \sin 2\theta_{12} \left(\frac{\delta m^2 x}{4E}\right)$

$B = s_{23} \sin 2\theta_{13} e^{i\delta} e^{-i\frac{\Delta m^2 x}{4E}} \sin\left(\frac{\Delta m^2 x}{4E}\right)$

Notice that:

- if $P_{interf} < 0$ then $|P_{interf}| \leq P_{sol} + P_{atm}$, otherwise $|A+B|^2 < 0$
- if $P_{interf} > 0$ then $P_{interf} \leq P_{sol} + P_{atm}$, otherwise $|A-B|^2 < 0$

Therefore: $|P_{interf}| \leq P_{sol} + P_{atm}$ always (never dominates)

- P_{sol} dominates for $\theta_{13} \to 0$
- P_{atm} dominates for $\delta m^2 \to 0$

One of the most widely studied problems, which makes use of the previous expressions for $P_{e\mu}$ (and the analogous for $\bar{\nu}$) is the so-called DEGENERACY :

measurements of $P_{e\mu}$ do not uniquely determine the unknowns δ and θ_{13}

→ need to make multiple measurements (at different L, E, etc...) to "break the degeneracy"

Interest is great because experiments will be costly (and thus must be optimized)

END