Dottorato di Ricerca Bari 2006

Physics of massive ν_s

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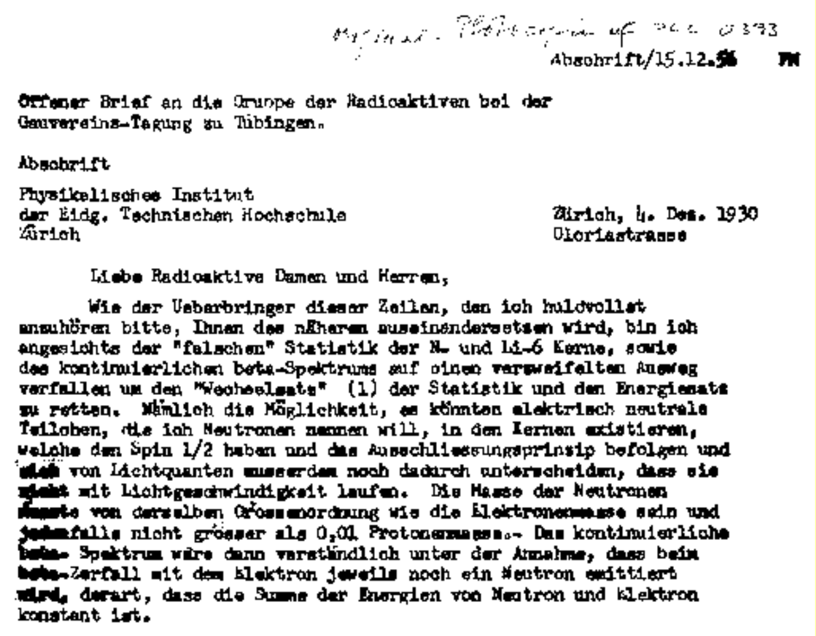
LECTURE I (1st part)

A brief introduction

ν

The oldest **fundamental** particle after the electron and the photon (Pauli, 1930)



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Abschrift/15.12.56 PM

Offener Brief an die Gruppe der Radioaktiven bei der Gauvereins-Tagung zu Tübingen.

Abschrift

Physikalisches Institut
der Eidg. Technischen Hochschule
Zürich

Zürich, 4. Dez. 1930
Gloriastrasse

Liebe Radioaktive Damen und Herren,

Wie der Ueberbringer dieser Zeilen, den ich huldvollst anzuhören bitte, Ihnen des näheren auseinandersetzen wird, bin ich angesichts der "falschen" Statistik der N- und Li-6 Kerne, sowie des kontinuierlichen beta-Spektrums auf einen verzweifelten Ausweg verfallen um den "Wechselsatz" (1) der Statistik und den Energiesatz zu retten. Nämlich die Möglichkeit, es könnten elektrisch neutrale Teilchen, die ich Neutronen nennen will, in den Kernen existieren, welche den Spin 1/2 haben und das Ausschliessungsprinzip befolgen und sich von Lichtquanten ausserdem noch dadurch unterscheiden, dass sie nicht mit Lichtgeschwindigkeit laufen. Die Masse der Neutronen müsste von derselben Grössenordnung wie die Elektronenmasse sein und jedenfalls nicht grösser als 0,01 Protonenmasse.- Das kontinuierliche beta- Spektrum wäre dann verständlich unter der Annahme, dass beim beta-Zerfall mit dem Elektron jeweils noch ein Neutron emittiert wird, derart, dass die Summe der Energien von Neutron und Elektron konstant ist.

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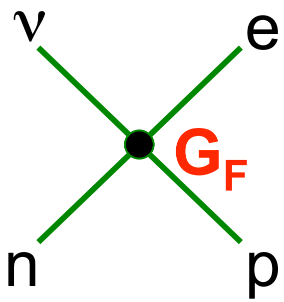
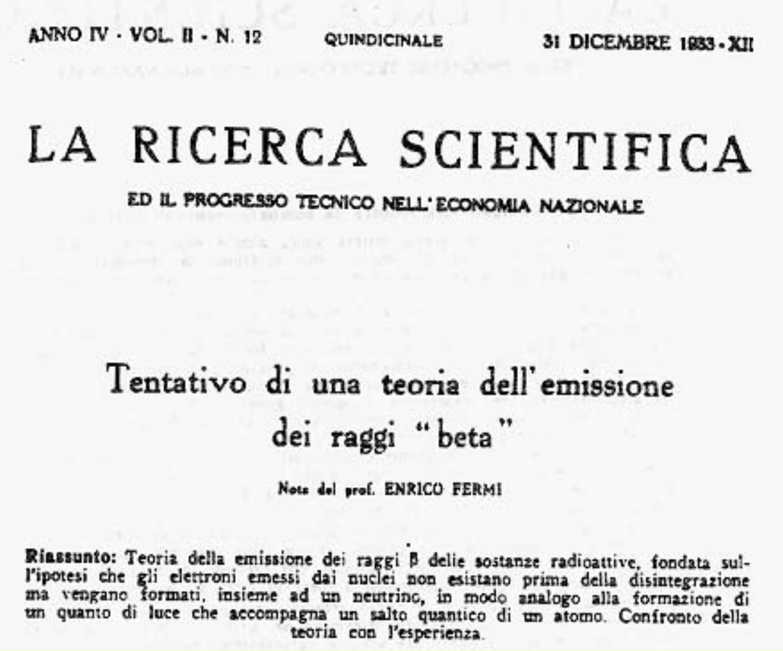
First kinematical properties: spin 1/2, small mass, no charge



Baptised and quantized within four-fermion effective interaction (Fermi, 1933-34)



ANNO IV - VOL. II - N. 12 QUINDICINALE 31 DICEMBRE 1933 - XII

LA RICERCA SCIENTIFICA

ED IL PROGRESSO TECNICO NELL'ECONOMIA NAZIONALE

Tentativo di una teoria dell'emissione dei raggi "beta"

Nota del prof. ENRICO FERMI

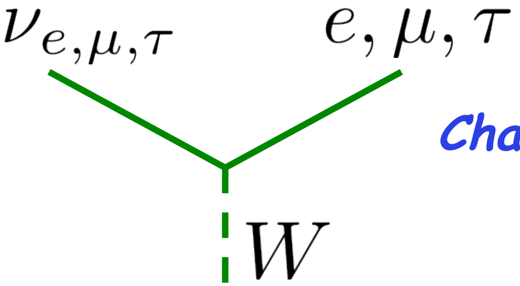
Riassunto: Teoria della emissione dei raggi β delle sostanze radioattive, fondata sull'ipotesi che gli elettroni emessi dai nuclei non esistano prima della disintegrazione ma vengano formati, insieme ad un neutrino, in modo analogo alla formazione di un quanto di luce che accompagna un salto quantico di un atomo. Confronto della teoria con l'esperienza.

First dynamical properties: Weak interactions, Fermi constant

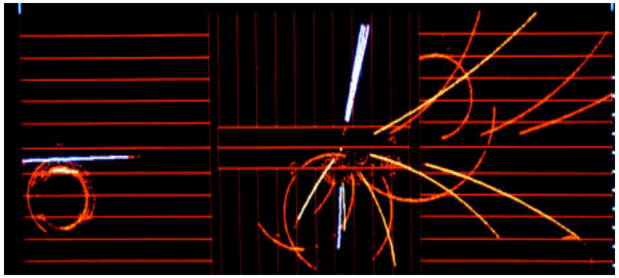
After > 70 years of research we have learned a lot more, e.g., that **neutrinos come in three flavors,**

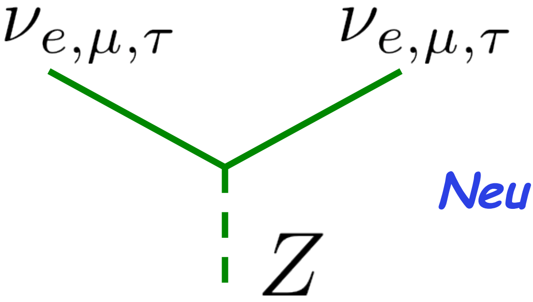
$$\begin{pmatrix} \nu_e \\ e \end{pmatrix} \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} \quad \begin{matrix} \leftarrow & q = 0 \\ \leftarrow & q = -1 \end{matrix} \quad (\Delta q = 1)$$

and that the Fermi interaction is mediated by a charged vector boson **W**, with a neutral counterpart: the vector boson **Z**

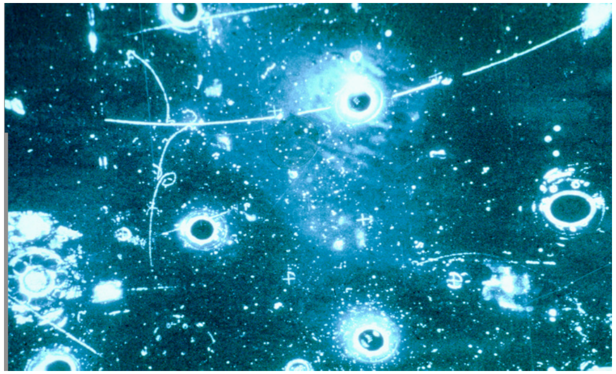


Charged current (Δq=1)





Neutral current (Δq=0)



Despite great progress, only recently we have got (or can reasonably hope to get "soon") an answer to some fundamental questions asked in the last century:

How small is the neutrino mass ?

(Pauli, Fermi, '30s)

Is the neutrino its own antiparticle?

(Majorana, '30s)

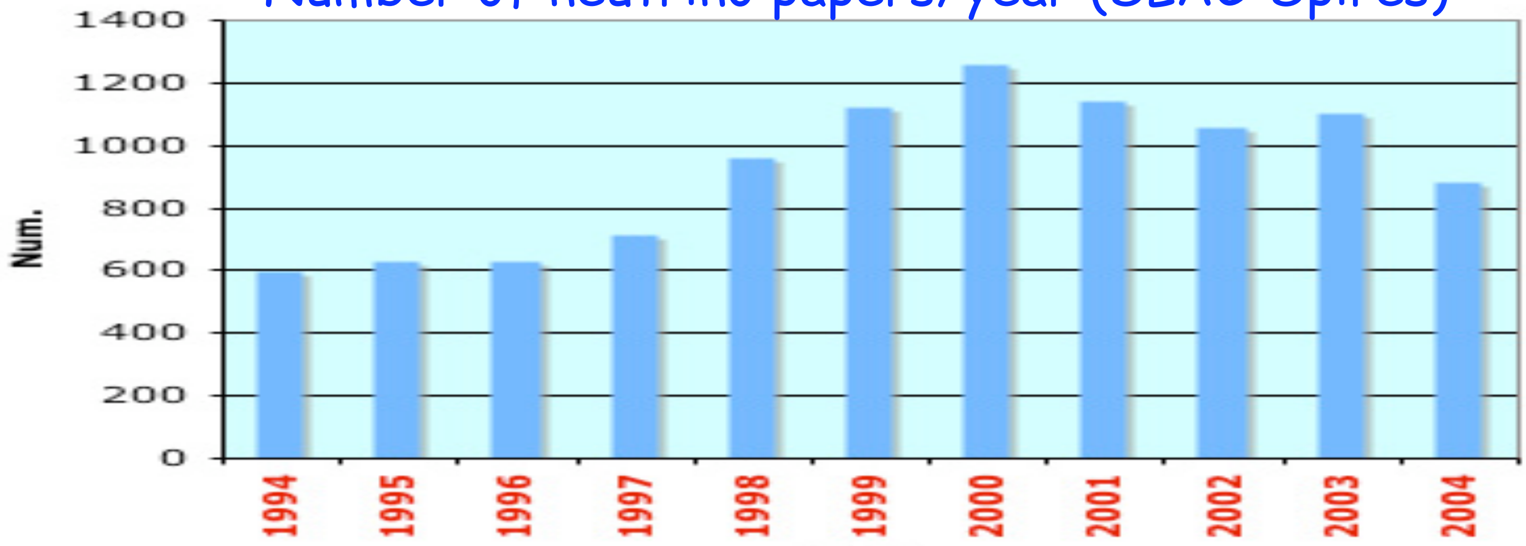
Do ν_s of different flavors trasform ("oscillate") among them?

(Pontecorvo, Maki-Nakagawa-Sakata, '60s)

In particular, one can give an affirmative -and rather detailed- answer to the last question. Explosion of interest (both expt. and theor.)

O(10^4) neutrino papers in the last decade. Boost after 1998 (evidence for atmospheric ν oscillations)

Number of neutrino papers/year (SLAC-Spires)



Many excellent neutrino reviews and books exist. Ask me for refs. or browse the "ν unbound" website: www.nu.to.infn.it

Hereafter, I will only touch a few selected topics, and cite literature only occasionally (some Refs will be given at the end)

ν interactions & masses:

elements of theory

Fermion currents in the Standard Model $SU(2)_L \times U(1)_Y$

Building blocks:

$$\begin{pmatrix} U^\alpha \\ D^\alpha \end{pmatrix}_L \qquad U^\alpha_R \qquad D^\alpha_R$$

U = "up" fermions

D = "down" fermions

$\alpha = 1, 2, 3$ = generation index

$$P_{L,R} = \frac{1 \mp \gamma_5}{2}$$

Charges:

$(T_\pm, T_3)$ = $SU(2)_L$ charges

$Y = 2(Q - T_3) = U(1)_Y$ charge

Q = e.m. charge

Gauge bosons (after SSB):

$W^\pm_\mu$ $(m = M_W)$

Z_μ $(m = M_Z)$

A_μ $(m = 0)$

Fermion currents:

$$W^\pm_\mu \left\{ \begin{array}{l} J^+_\mu = \sum_\alpha \bar{D}^\alpha_L \gamma_\mu U^\alpha_L \\ J^-_\mu = \sum_\alpha \bar{U}^\alpha_L \gamma_\mu D^\alpha_L \end{array} \right.$$

$$Z_\mu \left\{ \begin{array}{l} J^Z_\mu = \sum_\alpha \bar{U}^\alpha_L (T_3 - Q\sin^2\theta_W)\gamma_\mu U^\alpha_L \\ \qquad + U^\alpha_R(-Q\sin^2\theta_W)\gamma_\mu U^\alpha_R + (U\to D) \end{array} \right.$$

$$A_\mu \left\{ J^{EM}_\mu = \sum_\alpha \bar{U}^\alpha Q \gamma_\mu U^\alpha + (U \to D) \right.$$

Low-energy limit:

$$\mathcal{L}_{CC+NC} = -\frac{4G_F}{\sqrt{2}}\left[J^+_\mu J^-_\mu + \rho J^Z_\mu J^{\mu Z}\right]$$

$\rho = 1$ if SSB induced by Higgs doublet

$\tan\theta_W = g'/g$

$g = SU(2)_L$ coupling

$g' = U1(Y)_L$ "

θ_W = bookkeeping parameter
(can be eliminated in terms of mass spectrum + (α, G_F) + α_s)

Probing fermion currents with neutrinos

Neutrinos have been used to:

1) Assess strength of weak inter. (G_F)
2) Probe V-A structure of $J_\mu^\pm$ (CC)
3) Probe $(T_3 - Qs_w^2)$ charge of J_μ^Z (NC)
4) Probe CC+NC interference

…

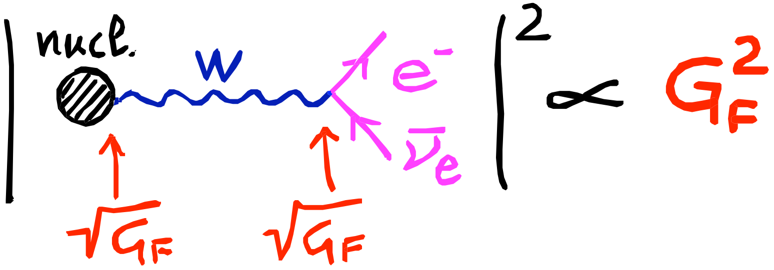
Examples:

1) β-decay, μ decay
2) $\pi \to \mu\bar{\nu}, e\bar{\nu}$ decay
3) $\overset{(-)}{\nu}_\mu e$ scattering
4) $\overset{(-)}{\nu}_e e$ scattering

1) Probing G_F in beta-decay and muon decay

β-decay

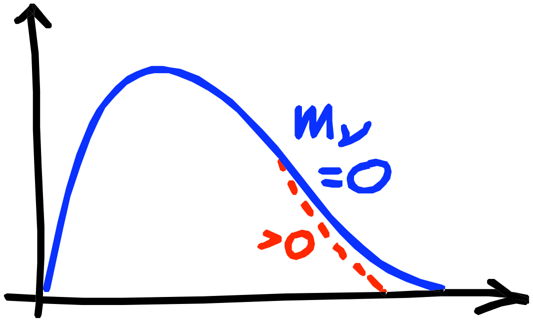
rate: $d\Gamma \propto G_F^2 \times (\text{phase sp.})$



energy spectrum:

$$\frac{d\Gamma}{dE_e} \propto G_F^2 p_e E_e (Q-E_e)^2 \qquad (m_\nu \equiv 0)$$

$$G_F^2 p_e E_e (Q-E_e)\sqrt{(Q-E_e)^2+m_\nu^2} \qquad (>0)$$



μ-decay

$$\Gamma_\mu = \frac{1}{\tau_\mu} \propto G_F^2 m_\mu^5$$

"defines" G_F

2) Probing V-A structure in pion decay

Dirac eq. for free particle (Weyl repres.)

$$\begin{bmatrix} -\frac{m}{E} & 1+h \\ 1-h & -\frac{m}{E} \end{bmatrix} \begin{bmatrix} \phi_R \\ \phi_L \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

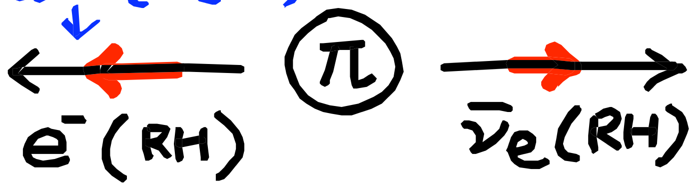
$$h = \frac{\vec{p}\cdot\vec{\sigma}}{E} \quad \leftarrow \text{helicity}$$

$$\phi_{R,L} = \frac{1 \pm \gamma_5}{2} \quad \leftarrow \text{chirality}$$

For $m/E \to 0$: helicity $\simeq$ chirality

$$h\,\phi_{R,L} \simeq \pm\phi_{R,L} + O(m/E)$$

"Wrong" chirality up to $O(m_e/E)$



$\pi \to e^- \bar{\nu}_e$ forbidden by V-A for $m_e \to 0$

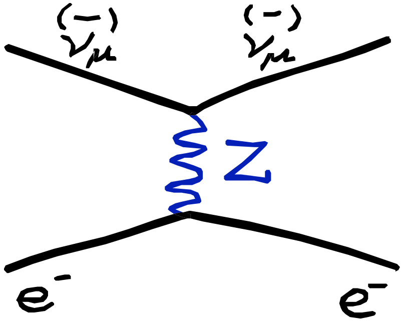
$$\frac{\Gamma(\pi \to e\nu)}{\Gamma(\pi \to \mu\nu)} \simeq$$

$$\simeq \underbrace{\left(\frac{m_e}{m_\mu}\right)^2}_{\ll 1} \underbrace{\left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2}_{>1} \ll 1$$

$\ll 1$ chirally suppressed

> 1 phase space

3) Probing (T_3-$Q\sin^2\theta_W$) NC structure with neutrinos



$\overset{(-)}{\nu}_\mu$ scattering on electrons

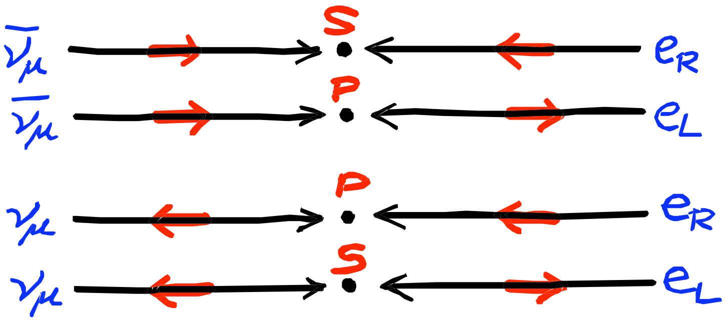
NC electron charges:

$\epsilon_L = (T_3 - Qs_W^2)e_L = -\frac{1}{2} + s_W^2$

$\epsilon_R = (T_3 - Qs_W^2)e_R = 0 + s_W^2$

At high energy, helicity ~ chirality and total (ν,e) spin J=0 (S-wave) or J=1 (p-wave) in C.M. system

$d\sigma/dy$ $\quad$ $\left(y = \frac{E_e}{E_\nu}\right)$



$\propto \epsilon_R^2$

$\epsilon_L^2(1-y)^2$

$\epsilon_R^2(1-y)^2$

ϵ_L^2

At low energy, helicity ≠ chirality and a further LR correction appear

$\propto \epsilon_L \epsilon_R \frac{m_e}{E_\nu} \cdot y$

Differential cross sections:

$$\frac{d\sigma}{dy}(\bar{\nu}_\mu e^-) \simeq \frac{2G_F^2 m_e E_\nu}{\pi}\left(\epsilon_R^2+\epsilon_L^2(1-y)^2\right)$$

$$\frac{d\sigma}{dy}(\nu_\mu e^-) \simeq \frac{2G_F^2 m_e E_\nu}{\pi}\left(\epsilon_L^2+\epsilon_R^2(1-y)^2\right)$$

Total cross sections:

$\int(1-y)^2 dy = 1/3 \leftarrow$ only 1/3 of $\vec{J}=1$ states allowed by J conservat.

$$\sigma(\bar{\nu}_\mu e^-) \propto \left(\epsilon_R^2+\tfrac{1}{3}\epsilon_L^2\right)$$

$$\sigma(\nu_\mu e^-) \propto \left(\epsilon_L^2+\tfrac{1}{3}\epsilon_R^2\right)$$

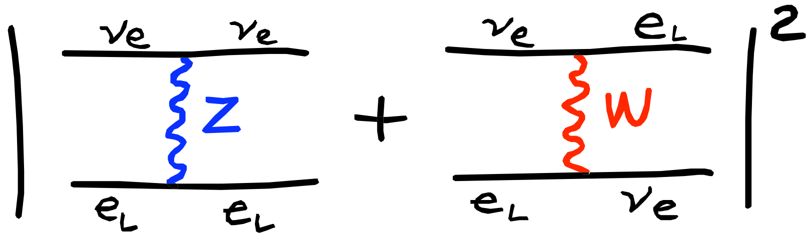
"History":

$$R=\frac{\sigma(\nu)}{\sigma(\bar{\nu})}=\frac{3\epsilon_L^2+\epsilon_R^2}{3\epsilon_R^2+\epsilon_L^2}=3\,\frac{1-4s_W^2+\frac{16}{3}s_W^4}{1-4s_W^2+16s_W^4}$$

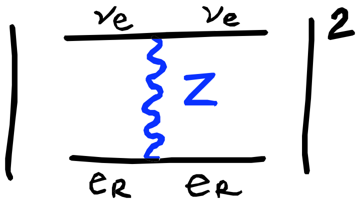
allowed first estimates of s_W^2 and of tree-level M_W and M_Z from:

$$s_W^2=\pi\alpha/\sqrt{2}\,G_F M_W^2 \quad ; \quad s_W^2=1-M_W^2/M_Z^2$$

4) Probing W-Z interference with neutrinos



$$\propto (\epsilon_L + 1)^2$$



$$\propto \epsilon_R^2 (1-y)^2$$

$$\frac{d\sigma}{dy}(\nu_e e^-) \simeq \frac{2G_F^2 m_e E_\nu}{\pi}\left[(\epsilon_L + 1)^2 + \epsilon_R^2 (1-y)^2\right]$$

$$\frac{d\sigma}{dy}(\bar{\nu}_e e^-) \simeq \frac{2G_F^2 m_e E_\nu}{\pi}\left[(\epsilon_R + 1)^2 + \epsilon_L^2 (1-y)^2\right]$$

W-Z INTERFERENCE

Implications →

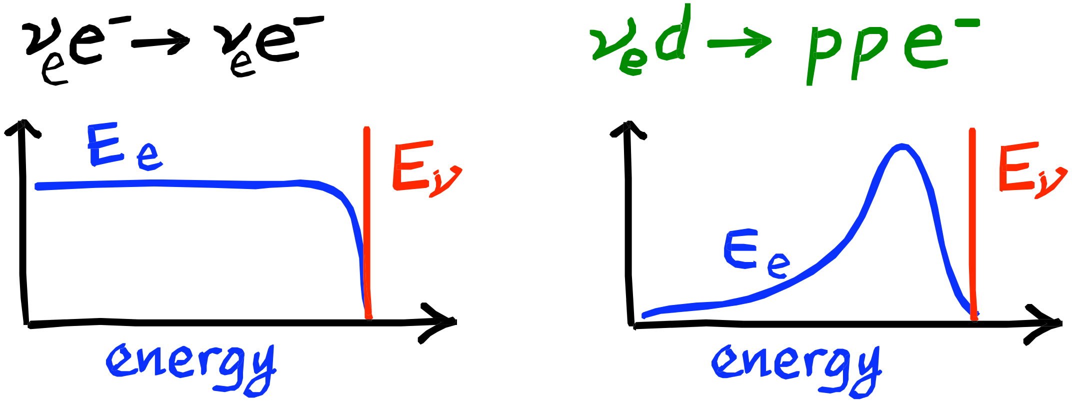
- $\sigma(\nu_\mu) < \sigma(\nu_e)$

$$\frac{\sigma(\nu_\mu e)}{\sigma(\nu_e e)} \simeq \frac{\epsilon_L^2 + \epsilon_R^2/3}{(\epsilon_L+1)^2 + \epsilon_R^2/3} \simeq \frac{1}{7}$$

- $\nu_e e^- \to \nu_e e^-$: flat spectrum

$$\frac{d\sigma}{dy}(\nu_e e^-) \propto 1 + \underbrace{\frac{\epsilon_R^2}{(\epsilon_L+1)^2}}_{\text{small}}(1-y)^2 \sim \text{const}$$

...to be compared with



Important for solar ν experiments

Fermion masses in the Standard Model

$$\phi_{\text{Higgs}} = \begin{pmatrix}\phi^+\\ \phi^0\end{pmatrix} \xrightarrow{SSB} \begin{pmatrix}0\\ v/\sqrt{2}\end{pmatrix}$$

Yukawa lagrangian:

$$-\mathcal{L}_Y = \sum_{\alpha\beta} f_D^{\alpha\beta}\,\overline{(U^\alpha, D^\alpha)}_L \begin{pmatrix}0\\ v/\sqrt{2}\end{pmatrix} D_R^\beta$$

$$+ \sum_{\alpha\beta} f_U^{\alpha\beta}\,\overline{(U^\alpha, D^\alpha)}_L \begin{pmatrix}v/\sqrt{2}\\ 0\end{pmatrix} U_R^\beta \quad \leftarrow \text{Get mass from } \tilde{\Phi} = i\sigma_2\phi^*$$

$$= \sum_{\alpha\beta} \bar{D}_L^\alpha \underset{\uparrow}{M_D^{\alpha\beta}} D_R^\beta + \bar{U}_L^\alpha \underset{\uparrow}{M_U^{\alpha\beta}} U_R^\beta$$

Generic 3×3 complex matrices

→ Diagonalization

Theorem: Generic M $(N\times N)$ is diagonalizable through biunitary transformation: $S^{+}MT=M_d$

where $M_d=\mathrm{diag}(m_1,m_2,\dots,m_N)$

and $SS^{+}=1=TT^{+}$

Proof: MM^{+} is hermitian

$$\rightarrow S^{+}(MM^{+})S=M_d^2=\mathrm{diag}(m_1^2,\dots,m_N^2)$$

with $m_i^2=(M_d^2)_{ii}=[(S^{+}M)(S^{+}M)^{+}]_{ii}$

$$=\sum_j (S^{+}M)_{ij}(S^{+}M)^{*}_{ij}$$

$$=\sum_j |S^{+}M|^2_{ij}>0$$

$\rightarrow MM^{+}$ has real, positive eigenvalues m_i^2

Define $M_d=\sqrt{M_d^2}=\mathrm{diag}(m_1,m_2,\cdots,m_N)$

Then: $H=SM_dS^{+}$ ← hermitian

$V=H^{-1}M$ ← unitary

$T=V^{+}S$ ← unitary

$$M_d=S^{+}HS=S^{+}MV^{+}S=S^{+}MT$$

Invariance: the currents

$$J_\mu^- = \sum_\alpha \bar{U}_L^\alpha \gamma_\mu D_L^\alpha$$

$$J_\mu^Z = \sum_\alpha \bar{U}_L^\alpha (T_3 - Q s_W^2) \gamma_\mu U_L^\alpha + \bar{U}_R^\alpha (-Q s_W^2) \gamma_\mu U_R^\alpha + (U \to D)$$

$$J_\mu^{EM} = \sum_\alpha \bar{U}^\alpha Q \gamma_\mu U^\alpha + (U \to D)$$

are invariant under the transformations

$$\begin{aligned} &(i) \quad U_R^\alpha \to T^{\alpha\beta} U_R^\beta \\ &(ii) \quad U_L^\alpha \to S^{\alpha\beta} U_L^\beta \\ &(iii) \quad D_L^\alpha \to S^{\alpha\beta} D_L^\beta \\ &(iv) \quad D_R^\alpha \to W^{\alpha\beta} D_R^\beta \end{aligned}$$

(ii), (iii): same S

$SS^\dagger = 1$

$TT^\dagger = 1$

$WW^\dagger = 1$

This fact implies that either M_U or M_D can be diagonalized without affecting currents

Usual "trick" for quarks:

Use properties (i), (ii), (iii) to identify T and S with the matrices diagonalizing M_U: $M_U = S^{+} M_U^{diag} T$

Then use (iv) to identify W with one of the matrices diagonalizing M_D:

$M_D = V^{+} M_D^{diag} W$ → only V physical:

$$D_L^{\alpha} \longrightarrow V^{\alpha\beta} D_L^{\beta}$$

The V-transformation affects $J_\mu^{\pm}$ (but not J_μ^{EM}, J_μ^{Z}):

$$J_\mu^{-} \rightarrow \sum_{\alpha\beta} \bar{U}_L^{\alpha} \gamma_\mu \underset{\substack{\uparrow \\ \text{CKM matrix}}}{V^{\alpha\beta}} D_L^{\beta}$$

What about leptons ?

If $m_\nu = 0$ (no ν_R) then only 1 matrix needed for diagonalization → no observable CKM lepton matrix

If we introduce ν_R^α in the same way as for quarks then ...

can get $m_\nu > 0$ but...

- no hint for smallness of m_ν
- mass terms for ν can be more general than for quarks since ν is neutral

Massless and massive (neutral) fermions

In Dirac representation:

$$\gamma^0_D = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad \vec{\gamma}_D = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix} \quad \gamma^5 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

"PARTICLE" solution $\psi_P \sim \begin{bmatrix} \xi \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\xi \end{bmatrix} e^{-ip_\mu x^\mu}$ $\xi\xi^+ = 1$

"ANTIPARTICLE" solution $\psi_A \sim \begin{bmatrix} \frac{\sigma\cdot p}{E+m}\chi \\ \chi \end{bmatrix} e^{ip_\mu x^\mu}$ $\chi\chi^+ = 1$

ξ, χ = Pauli spinors

Nonrelativistic (particle) limit : $\psi_P \sim \begin{bmatrix} \xi \\ 0 \end{bmatrix}$ $\overline{\psi}_P \sim \begin{bmatrix} \xi^+ \\ 0 \end{bmatrix}^T$

$$S = \overline{\psi}\psi \simeq |\xi|^2$$

$$P = \overline{\psi}\gamma^5\psi \simeq 0$$

$$\left.\begin{aligned} V &= \overline{\psi}\gamma^\mu\psi \simeq (|\xi|^2, \vec{0}) \\ A &= \overline{\psi}\gamma^\mu\gamma^5\psi \simeq (0, \xi^+\vec{\sigma}\xi) \end{aligned}\right\} \text{useful later}$$

Dirac representation useful to define the particle-antiparticle conjugation operator $\mathcal{C}$

$$\psi^c = \mathcal{C}(\psi)$$
$$\psi_{P,A} = \mathcal{C}(\psi_{A,P})$$

$$\begin{aligned}\mathcal{C}(\psi) &= i\gamma^2\psi^* \\ &= i\gamma^2\gamma^0\overline{\psi}^T \\ &= C\,\overline{\psi}^T \\ &= \psi^c\end{aligned}$$

$$C = i\gamma^2\gamma^0$$

1) Prove that $\mathcal{C}(\psi_P) = \psi_A$; hint : use $\sigma_2\vec{\sigma}^* = -\vec{\sigma}\sigma_2$ and set $\chi = -i\sigma_2\xi^*$

2) Prove that if ψ is e.m. charged, $[i\gamma^\mu(\partial_\mu - iqA_\mu) - m]\psi = 0$, then $[i\gamma^\mu(\partial_\mu + iqA_\mu) - m]\psi^c = 0$

Convention :

When operations such as $P_{L,R}$, $\overline{(-)}$, and $\mathcal{C}$, are involved :

$P_{L,R}$ acts before C which acts before $\overline{(\cdot)}$

$\Downarrow$

$$\psi^c_{L,R} = (P_{L,R}\psi)^c = (\psi_{L,R})^c = P_{R,L}(\psi^c)$$

$$\overline{\psi}_{L,R} = \overline{(P_{L,R}\psi)} = \overline{(\psi_{L,R})} = \overline{\psi}\, P_{R,L}$$

$$\overline{\psi}^c = \overline{(\psi^c)}$$

$$\overline{\psi}^c_{L,R} = \overline{(P_{L,R}\psi)^c} = \overline{(\psi_{L,R})^c} = \overline{P_{R,L}(\psi^c)} = \overline{\psi}^c P_{L,R}$$

Weyl representation and Lorentz group

Let's change basis (from "Dirac" to "Weyl"):

$$\psi \rightarrow T\psi$$

$$\gamma^\mu \rightarrow T\gamma^\mu T^{-1}$$

$$T = \frac{1}{\sqrt{2}}(\gamma^0_D + \gamma^5_D) = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ I & -I \end{bmatrix}$$

$$\gamma^0_W = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad \vec{\gamma}_W = \begin{bmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} \quad \gamma^5_W = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

Then:

$$\psi_R = \frac{1+\gamma_5}{2}\psi = \begin{bmatrix} \phi_R \\ 0 \end{bmatrix}$$

$$\psi_L = \frac{1-\gamma_5}{2}\psi = \begin{bmatrix} 0 \\ \phi_L \end{bmatrix}$$

"fundamental" objects under Lorentz group

If x^μ transforms as :

$$x'^{\mu} = e^{i(\underset{\substack{\uparrow \\ \text{ROTATION}}}{\vec{\omega}}\cdot\vec{J} + \underset{\substack{\uparrow \\ \text{BOOST}}}{\vec{u}}\cdot\vec{K})} x^{\mu}$$

then $\phi_{R,L}$ transform as :

$$\phi'_R = e^{i(\vec{\omega} - i\vec{u})\frac{\vec{\sigma}}{2}} \phi_R$$

$$\phi'_L = e^{i(\vec{\omega} + i\vec{u})\frac{\vec{\sigma}}{2}} \phi_L$$

} coupled by Dirac equation; decoupled only if $m=0$ (Weyl spinors)

Theorem : Given ϕ_R (RH), $i\sigma_2\phi_R^*$ is LH ; given ϕ_L (LH), $-i\sigma_2\phi_L^*$ is RH
(Hint: use $\sigma_2\vec{\sigma}^* = -\vec{\sigma}\sigma_2$ and infinitesimal transform.)

→ can build a Dirac spinor ψ from two RH spinors u & v : $\psi = \begin{bmatrix} u \\ i\sigma_2 v^* \end{bmatrix} = \begin{bmatrix} \psi_R \\ \psi_L \end{bmatrix}$

(or from two LH ones)

$\overset{(-)}{\psi}{}^{(c)}_{(L,R)}$ components in Weyl basis:

$$\psi = \begin{bmatrix} u \\ i\sigma_2 v^* \end{bmatrix} \qquad \psi_L = \begin{bmatrix} 0 \\ i\sigma_2 v^* \end{bmatrix} \qquad \psi_R = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

$$\bar{\psi} = [-iv^T\sigma_2, u^+] \qquad \bar{\psi}_L = [-iv^T\sigma_2, 0] \qquad \bar{\psi}_R = [0, u^+]$$

$$\psi^c = \begin{bmatrix} v \\ i\sigma_2 u^* \end{bmatrix} \qquad \psi^c_L = \begin{bmatrix} v \\ 0 \end{bmatrix} \qquad \psi^c_R = \begin{bmatrix} 0 \\ i\sigma_2 u^* \end{bmatrix}$$

$$\bar{\psi}^c = [-iu^T\sigma_2, v^+] \qquad \bar{\psi}^c_L = [0, v^+] \qquad \bar{\psi}^c_R = [-iu^T\sigma_2, 0]$$

...with $\mathcal{C}$ swapping $u \leftrightarrow v$

In general, no relation between u and v

Given $\psi = \begin{bmatrix} u \\ i\sigma_2 v^* \end{bmatrix}$ $(u, v$ R.H.$)$:

$u \neq v \rightarrow$ Dirac ν

$u = v \rightarrow$ Majorana ν

For Majorana neutrinos, $u = v$ implies that $\psi = \psi^c$ (see previous slide)

→ Majorana ν are their own antiparticles

→ They must be completely neutral (no e.m. charge, no generalized charge)

More generally, for Majorana ν's :

$$\psi_M = \psi_M^c \cdot e^{i\varphi_M}$$

← "Majorana creation phase" can be different from +1 (examples later)

Summary of ν representations:

$m=0$ Weyl	$\psi = \begin{bmatrix} \nu_R \\ 0 \end{bmatrix} = \psi_R$ or: $\psi = \begin{bmatrix} 0 \\ \nu_L \end{bmatrix} = \psi_L$	simplest massless case, 2 dof
$m \neq 0$ Major.	$\psi = \begin{bmatrix} \nu_R \\ i\sigma_2 \nu_R^* \end{bmatrix} = \psi_R + \psi_R^c = \psi^c$ or: $\psi = \begin{bmatrix} -i\sigma_2 \nu_L^* \\ \nu_L \end{bmatrix} = \psi_L + \psi_L^c = \psi^c$	simplest massive case, 2 dof
$m \neq 0$ Dirac	$\psi = \begin{bmatrix} \nu_R \\ \nu_L \end{bmatrix} = \psi_R + \psi_L \neq \psi^c$	general massive case, 4 dof

Paradox and resolution:

Define ν_e as the neutral fermion produced in β^+ decay of some nucleus

Define $\bar{\nu}_e$ as the neutral fermion produced in β^- decay of some nucleus

Q. How can it be $\nu_e = \bar{\nu}_e$ if:

$$\nu_e + n \longrightarrow p + e^-$$
$$\bar{\nu}_e + n \not\longrightarrow p + e^-$$
$$\bar{\nu}_e + p \longrightarrow n + e^+$$
$$\nu_e + p \not\longrightarrow n + e^+$$

?

A(1). Indeed, $\nu_e \neq \bar{\nu}_e$ (Dirac case) → Lepton number is conserved: $\Delta L_e = 0$

A(2). It is $\nu = \bar{\nu}$ (Majorana) and we are naming:

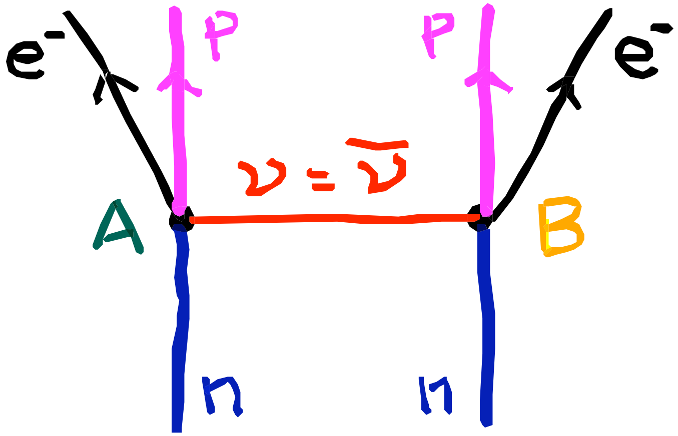
$$\begin{cases} \text{“}\nu_e\text{”} = P_L \nu \\ \text{“}\bar{\nu}_e\text{”} = P_R \nu \end{cases}$$

The initial "ν_e" is produced LH in β^+ decay and remains so up to $O(m/E)$. The reaction $\nu_e p \to n e^+$ is thus chirally suppressed by V-A

However, $O(m/E)$ does not mean "never". Such reaction can take place at small energies: $\Delta L_e = 2$ at $O(m/E)$

Majorana neutrinos and neutrinoless 2β decay

$0\nu 2\beta$ decay : a low-energy and extremely rare ([weak]^2!) reaction. A nucleus changes charge by two units and emits a couple of electrons :

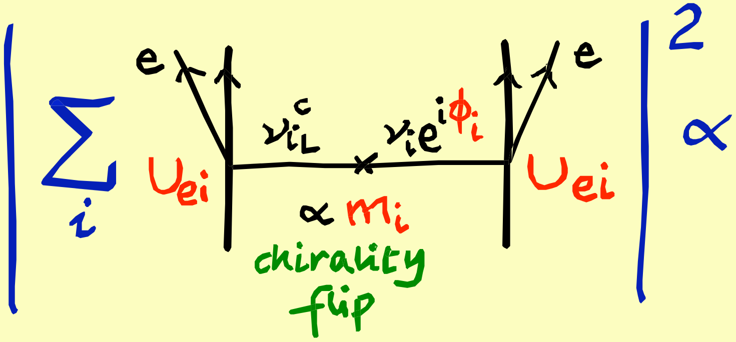


Intuitive picture :

- A $\overline{\nu}_e$ (RH) is emitted in A
- If it is massive, at O(m/E) it develops a LH component ← not possible for Weyl ν
- If $\nu = \overline{\nu}$, such component is a LH neutrino ← not possible for Dirac ν
- The ν_L is absorbed in B and an electron is emitted
- Init. state: no electrons; final stat: 2 electrons → $\Delta L_e = 2$ ← $0\nu 2\beta$ and $\Delta L_e = 2$ only possible for Majorana ν

Relevant parameter in $0\nu 2\beta$

In general, ν_e = superposition of Majorana fields ν_i with masses m_i, coefficients U_{ei}, and creation phases $\exp(i\phi_i)$



$$\propto \left|\sum_i U_{ei}^2 m_i e^{i\phi_i}\right|^2 = \left|\sum_i |U_{ei}|^2 m_i e^{i\phi_i'}\right|$$

$$= \langle m_{ee}\rangle^2 \quad \text{or}$$

$$= m_{\beta\beta}^2$$

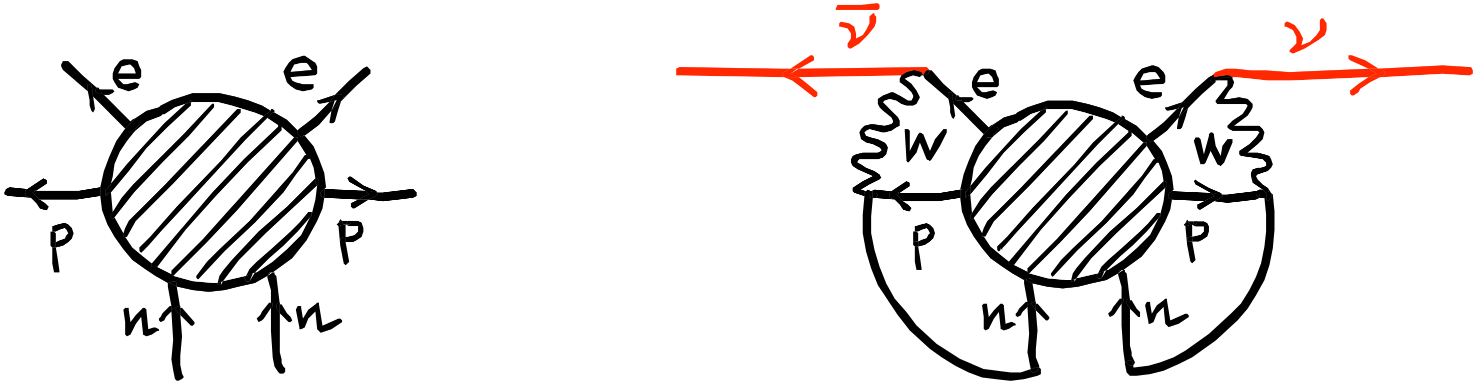
$m_{\beta\beta}$ = "effective Majorana mass"

Global phases ϕ_i' (mixing + Majorana) are physical → can get constructive or destructive interf. in (i,j) channels → $m_{\beta\beta}$ may be small due to "cancellations"

Deep link between $0\nu 2\beta$ decay and Majorana ν

Independently of the mechanism for $0\nu 2\beta$ decay…

… get a Majorana neutrino mass term if $0\nu 2\beta$ occurs



Neutrino mass terms for ONE FAMILY

Can generate $m\bar{\psi}\psi$ in 3 possible ways:

1) $\psi = \psi_L + \psi_R$ (Dirac) $\rightarrow \bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$
2) $\psi = \psi_L + \psi_L^c$ (Major.) $\rightarrow \bar{\psi}\psi = \bar{\psi}_L\psi_L^c + \bar{\psi}_L^c\psi_L$
3) $\psi = \psi_R + \psi_R^c$ (Major.) $\rightarrow \bar{\psi}\psi = \bar{\psi}_R\psi_R^c + \bar{\psi}_R^c\psi_R$

How? E.g. Higgs
- doublet Φ ← standard model Higgs
- triplet $\vec{\Phi}$ ← beyond standard model
- singlet φ ← " " "

$$\mathcal{L} \ni h(\bar{\nu}_L \bar{e}_L)\Phi\nu_R + h'(\bar{\nu}_L \bar{e}_L)\vec{\Phi}\cdot\frac{\vec{\sigma}}{2}\begin{pmatrix}\nu_L^c\\ e_L\end{pmatrix} + h''(\bar{\nu}_R^c\nu_R + \bar{\nu}_R\nu_R^c)\varphi$$

$\xrightarrow{\text{SSB}}$

$$m_D(\bar{\nu}_L\nu_R + \bar{\nu}_R\nu_L) \quad \text{Dirac}$$
$$+\, m_L(\bar{\nu}_L\nu_L^c + \bar{\nu}_L^c\nu_L) \quad \text{Major.}$$
$$+\, m_R(\bar{\nu}_R\nu_R^c + \bar{\nu}_R^c\nu_R) \quad \text{Major.}$$

doublet × doublet × singlet
doublet × triplet × doublet
singlet × singlet × singlet

Majorana mass terms not invariant under any global $U(1): \psi \rightarrow e^{i\phi}\psi$
→ no additive (lepton) number conserved

Mass lagrangian in matrix form

(Majorana basis)

$$-\mathcal{L}_m = \left(\bar{\nu}_L + \bar{\nu}_L^c, \bar{\nu}_R + \bar{\nu}_R^c\right) \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} \nu_L + \nu_L^c \\ \nu_R + \nu_R^c \end{pmatrix}$$

→ Intuitively clear that, in general, diagonalization will give Majorana ν as eigenstates (not Dirac ν)

Diagonalization exercise:

$$M = \begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix} \quad T = \mathrm{Tr}\, M = m_L + m_R$$

$$D = \det M = m_L m_R - m_D^2$$

Eigenvalues: $m_\pm = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4D}\right)$

$$\sin 2\theta = \frac{2 m_D}{\sqrt{T^2-4D}} \qquad \cos 2\theta = \frac{m_L - m_R}{\sqrt{T^2-4D}}$$

$$\begin{bmatrix} m_+ & 0 \\ 0 & m_- \end{bmatrix} = \begin{bmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix}$$

$$[v_1, v_2] \begin{bmatrix} m_L & m_D \\ m_D & m_R \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \underset{\text{eigenvec.}}{[v_1', v_2']} \begin{bmatrix} m_+ & 0 \\ 0 & m_- \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix}$$

$$\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

θ is <u>not</u> a "Cabibbo" angle (1 family only!)

"Dirac" case : $M = \begin{bmatrix} 0 & m \\ m & 0 \end{bmatrix}$

(recover Dirac ν)

- **Eigenvectors:**

$$\phi_1 = \frac{1}{\sqrt{2}}\left[(\nu_L+\nu_L^c)+(\nu_R+\nu_R^c)\right] \quad \text{mass } m$$

$$\phi_2 = \frac{1}{\sqrt{2}}\left[-(\nu_L+\nu_L^c)+(\nu_R+\nu_R^c)\right] \quad \text{mass } -m$$

- "Negative mass" not a problem (Majorana phase $= -1$). Define: which obeys Dirac eq. with $+m$

$$\tilde{\phi}_2 = \gamma_5\phi_2 = \frac{1}{\sqrt{2}}\left[(\nu_L-\nu_L^c)+(\nu_R-\nu_R^c)\right]$$

note: $\tilde{\phi}_2^c = -\tilde{\phi}_2$

- ϕ_1 and $\tilde{\phi}_2$ have both mass m. Observable (active) component is: → get a Dirac spinor $\nu (\neq \nu^c)$ with mass $m = m(\phi_1) = m(\phi_2)$

$$\nu_L = P_L\nu = P_L(\nu_L+\nu_R) = P_L\frac{1}{\sqrt{2}}(\phi_1+\tilde{\phi}_2)$$

$$= \frac{1}{\sqrt{2}}(\phi_1+\phi_2)_L$$

"See-saw" case $\mathcal{M} = \begin{bmatrix} 0 & m \\ m & M \end{bmatrix}$

- $\exists\, \nu_R$ in fermion multiplets of many SM extensions; e.g., $\underline{16}$ of SO(10):

  → get a Majorana mass term

  $$M(\bar{\nu}_R^c \nu_R + \bar{\nu}_R \nu_R^c)$$

  presumably large mass scale characterizing SM extension

$$\begin{pmatrix} u_L & u_L & u_L & \nu_L \\ d_L & d_L & d_L & e_L \\ u_R & u_R & u_R & \nu_R \\ d_R & d_R & d_R & e_R \end{pmatrix}$$

- Eigenvectors at $O(m/M)$:

$$\phi_1 = (\nu_R + \nu_R^c) + \frac{m}{M}(\nu_L + \nu_L^c)$$ ← heavy mass M

$$\phi_2 = (\nu_L + \nu_L^c) + \frac{m}{M}(\nu_R + \nu_R^c)$$ ← negative mass $-m^2/M$

$$\tilde{\phi}_2 = \gamma_5 \phi_2 = (\nu_L - \nu_L^c) + \frac{m}{M}(\nu_R - \nu_R^c)$$ ← light mass $+m^2/M$

- The light state is active $(\nu_L \in \tilde{\Phi}_2)$
- The light state mass can be very small (see-saw)

$$m(\Phi_2) = \frac{m^2}{M}$$

← Dirac scale (SSB)

← Beyond SM "heavy" scale

Neutrino masses for MORE FAMILIES

1) In the general case (Dirac + Majorana) start from:

- 3 LH gauge doublets $\nu_{\alpha L}$ $\alpha = e \mu \tau$
- n_s RH gauge singlets ν_{sR} $s = 1, 2, \dots, n_s$

2) Build column of LH fields

$$\nu_L = \begin{pmatrix} \nu_{\alpha L} \\ \nu^c_{sR} \end{pmatrix}$$

dim = $3 + n_s$

3) Write mass term: ... and diagonalize M

row of RH fields

$$\mathcal{L}_M = -\frac{1}{2} \overline{\nu_L^c} M \nu_L$$

column of LH fields

$$M = \begin{bmatrix} M_L & M_D \\ M_D' & M_R \end{bmatrix}$$

$M_L = 3 \times 3$ ← Majorana

$M_D^{(')} = 3 \times n_s$ ← Dirac

$M_R = n_s \times n_s$ ← Majorana

$\mathcal{L}$ antisimmetry + anticommutation rules →

$$\begin{cases} M_L = M_L^T \\ M_R = M_R^T \\ M_D' = M_D^T \end{cases}$$

→ $M = M^T$ (symmetric matrix)

Diagonalization in general (Dirac+Majorana) case

→ at least 3 important differences w.r.t. pure Dirac (quark-like) case

1) Eigenvectors ν_K (mass eigenstates) are generally Majorana → Expect $0\nu 2\beta$ decay

2) The LH column $\begin{pmatrix}\nu_{\alpha L}\\ \nu^c_{sR}\end{pmatrix}$ is a linear combination of ν_{kL} →

$$\begin{pmatrix}\nu_{eL}\\ \nu_{\mu L}\\ \nu_{\tau L}\\ \nu^c_{1R}\\ \nu^c_{2R}\\ \vdots\\ \nu^c_{n_s R}\end{pmatrix} = U \cdot \begin{pmatrix}\nu_{1L}\\ \nu_{2L}\\ \nu_{3L}\\ \vdots\\ \vdots\\ \vdots\\ \nu_{(3+n_s)L}\end{pmatrix}$$

Conversely, massive states are superpositions of active ν_α and sterile ν_s (active/sterile ν mixing)

3) Since M is symmetric, only one matrix needed for diagonaliz. (not biunitary) → less freedom to reabsorb phases

E.g. for 3 ν generations:

Dirac case ("quark-like") $U \ni \delta_{CP}$

Major. case $U \ni \delta_{CP}, \phi', \phi''$

RECAP

- Neutrino currents well understood and tested
- Neutrino nature (Weyl? Majorana? Dirac?) difficult to explore in practice, due to chirality of interactions and smallness of ν mass.
  However: $m_\nu \neq 0$ → not Weyl; $\exists\ 0\nu 2\beta$ → not Dirac
- Neutrino mass terms can be more general than in the quark sector, and point towards new physics in general
  - non standard Higgs sector
  - heavy RH scale
  - active-sterile mixing
  - ...

Dottorato di Ricerca Bari 2006

Physics of massive νs

Eligio Lisi, INFN, Bari, Italy

LECTURE I (2nd part)

Neutrino oscillations

- THEORY -

ν oscillations: general consequence of mixing of flavor states ν_α with massive states ν_β

$$\begin{array}{r} \text{3 active states} \left\{ \vphantom{\begin{matrix}a\\a\\a\end{matrix}} \right. \\ \text{sterile states} \left\{ \vphantom{\begin{matrix}a\\a\end{matrix}} \right. \end{array} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ \nu_s \\ \vdots \end{pmatrix} = U_{\alpha i} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \\ \vdots \end{pmatrix} \qquad (UU^\dagger = 1)$$

Smallness of ν mass splittings

→ macroscopic oscillation lengths

Smallness of neutrino masses
(w.r.t. to observable energies)

→ Can ignore exceedingly small chirality flips during propagation

→ Can use "Dirac-like" terminology
"ν" = ν_L , "$\bar{\nu}$" = ν_R, even for $\nu_{\text{Major.}}$

→ Can often treat ν's as "wavefunctions"
(and use QM-like notation)

Explore propagation Hamiltonians of increasing complexity
(especially in experimentally manageable flavor basis)

$$i\frac{\partial}{\partial t}\nu_\alpha = H_{\alpha\beta}\,\nu_\beta$$

3 massless ν in vacuum

$$i\frac{\partial}{\partial t}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = H\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} \qquad m(\nu_\alpha)=0$$

for a beam of momentum p :

$$H = \begin{bmatrix}E_e & & \\ & E_\mu & \\ & & E_\tau\end{bmatrix} = \begin{bmatrix}p & & \\ & p & \\ & & p\end{bmatrix} = p\cdot\mathbb{1}$$

$$|\nu_\alpha\rangle_t = e^{-ipt}|\nu_\alpha\rangle_0$$ ← flavor is conserved

Overall phase $\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} \rightarrow e^{i\phi}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}$ unobservable in squared amplitudes $|\langle\nu_\beta|\nu_\alpha\rangle|^2$

→ H defined mod. $\lambda\mathbb{1}$ in general

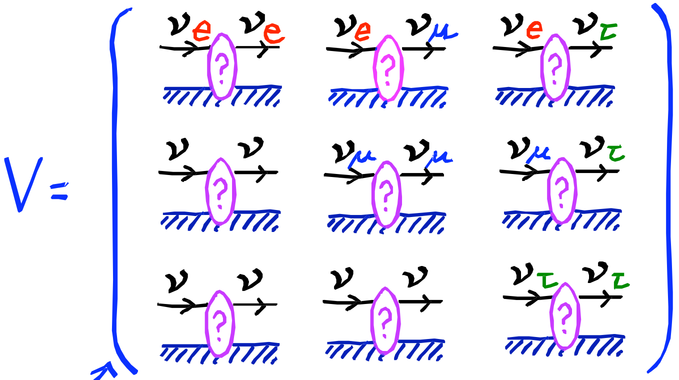
3 massless ν in matter

$i\frac{\partial}{\partial t}\nu_\alpha = H_{\alpha\beta}\nu_\beta$ $m(\nu_\alpha)=0$ $H = \underset{\text{kinematics}}{p\mathbb{1}} + \underset{\text{dynamics}}{V}$

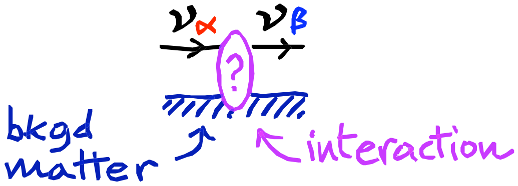
V = interaction energy in matter (Wolfenstein)

$$p\mathbb{1} = \begin{bmatrix} \xrightarrow{\nu_e} & 0 & 0 \\ 0 & \xrightarrow{\nu_\mu} & 0 \\ 0 & 0 & \xrightarrow{\nu_\tau} \end{bmatrix}$$

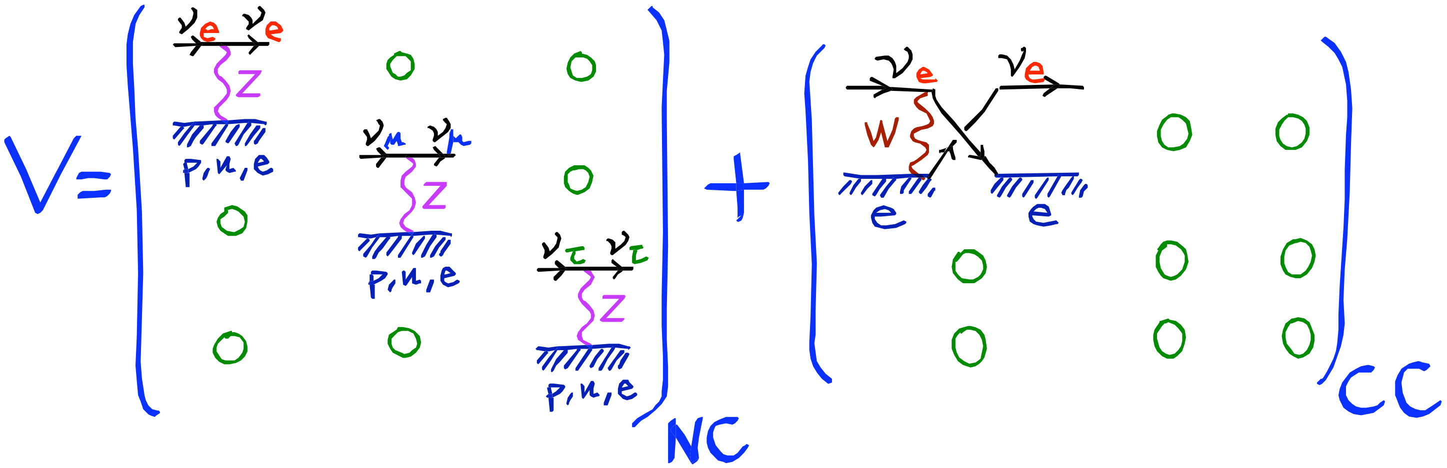
ν free streaming



dynamical contribution to forward scattering



Interaction "blob" well defined in standard EW model ⟶

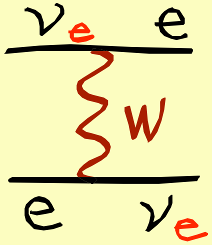


$$V = V_{NC} + V_{CC} \quad \text{with} \quad V_{NC} \propto \mathbb{1}$$

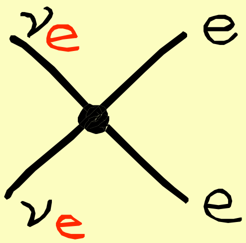
(up to small higher-order corrections)

→ Relevant term is the interaction energy difference V_{CC}

$V_{CC}^{ee} =$



$\simeq$



Evaluation of V_{CC}^{ee}

$$H_{CC} = \underbrace{\frac{G_F}{\sqrt{2}}\bar{e}\gamma^\mu(1-\gamma_5)\nu_e}_{J^{CC}} \otimes \underbrace{\bar{\nu}_e\gamma_\mu(1-\gamma_5)e}_{J^{CC}} \overset{\text{Fierz}}{=\!=} \underbrace{\frac{G_F}{\sqrt{2}}\bar{e}\gamma^\mu(1-\gamma_5)e}_{J_e} \; \underbrace{\bar{\nu}_e\gamma_\mu(1-\gamma_5)\nu_e}_{J_\nu}$$

From the ν viewpoint, the e^- is ~nonrelativistic and ~unpolarized

→ Dirac representation, $e \simeq \begin{bmatrix}\xi\\0\end{bmatrix}$

$$\bar{e}\gamma^\mu(1-\gamma_5)e \simeq (\underbrace{\xi^\dagger\xi}_{\substack{\text{density}\\ N_e}}, \underbrace{\xi^\dagger\vec{\sigma}\xi}_{\substack{\text{polarization}\\ \sim 0}}) \simeq N_e\delta_{\mu 0}$$

$$H_{CC} = \frac{G_F}{\sqrt{2}} N_e \bar{\nu}_e\gamma_0(1-\gamma_5)\nu_e = \underbrace{\sqrt{2}G_F N_e}_{\text{coupling}} \; \underbrace{\bar{\nu}_{eL}\gamma_0\nu_{eL}}_{\substack{\text{"static"}\\ \text{term}}}$$

$$\boxed{V_{CC}^{ee} = \sqrt{2}\, G_F N_e}$$

Exercise : prove that

$$\frac{A}{eV^2} = 1.526 \times 10^{-7} \left(\frac{N_e}{mol/cm^3}\right)\left(\frac{E}{MeV}\right)$$

where $A = 2EV = 2\sqrt{2}\, G_F N_e E$

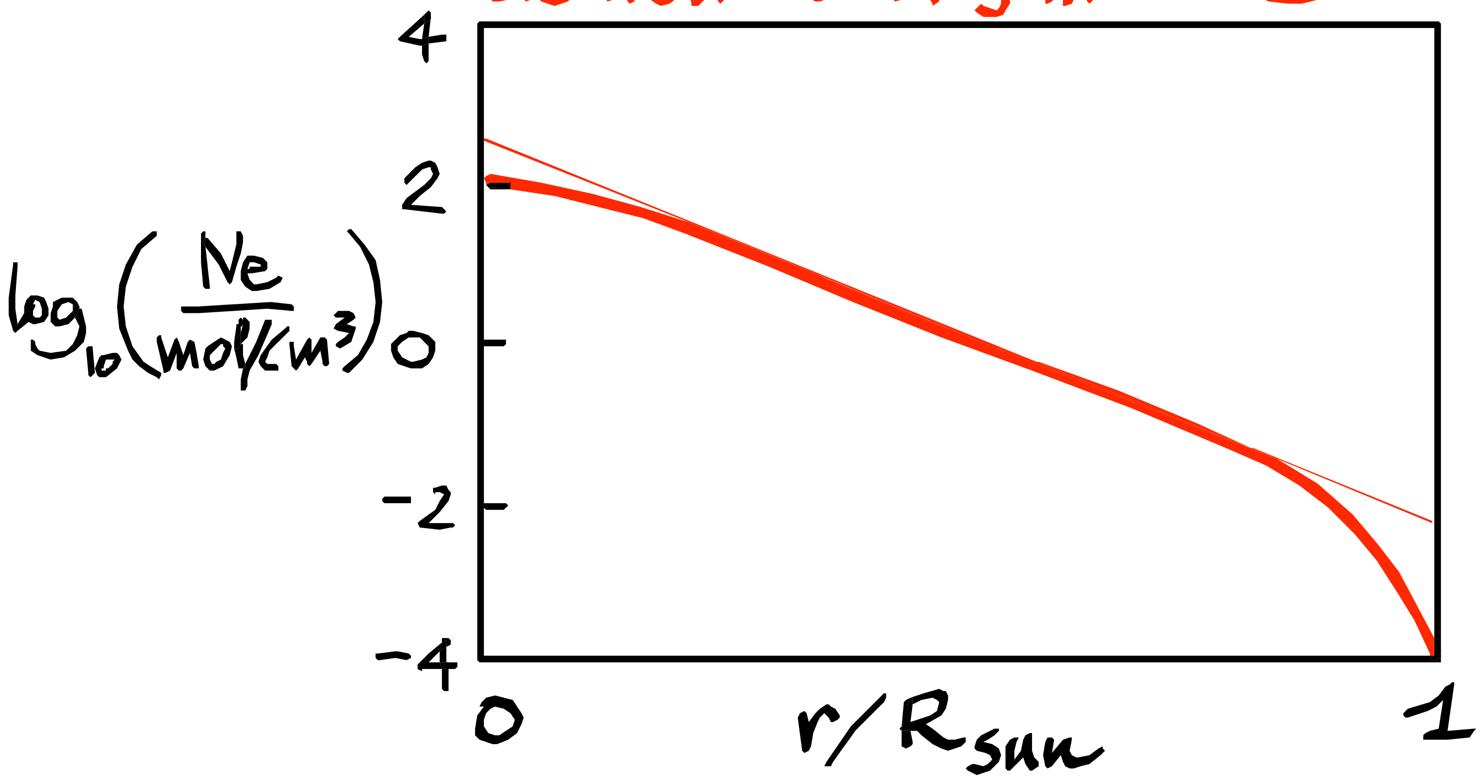
hint : remember that

$G_F = 1.16637 \times 10^{-5}\ GeV^{-2}$

and use (see tutorials)

$1 \frac{mol}{cm^3} = 4.627 \times 10^{-9}\ MeV^3$

electron density in the Sun



Exp. approximation:

$N_e \simeq N_e(0)\, e^{-x/r_0}$

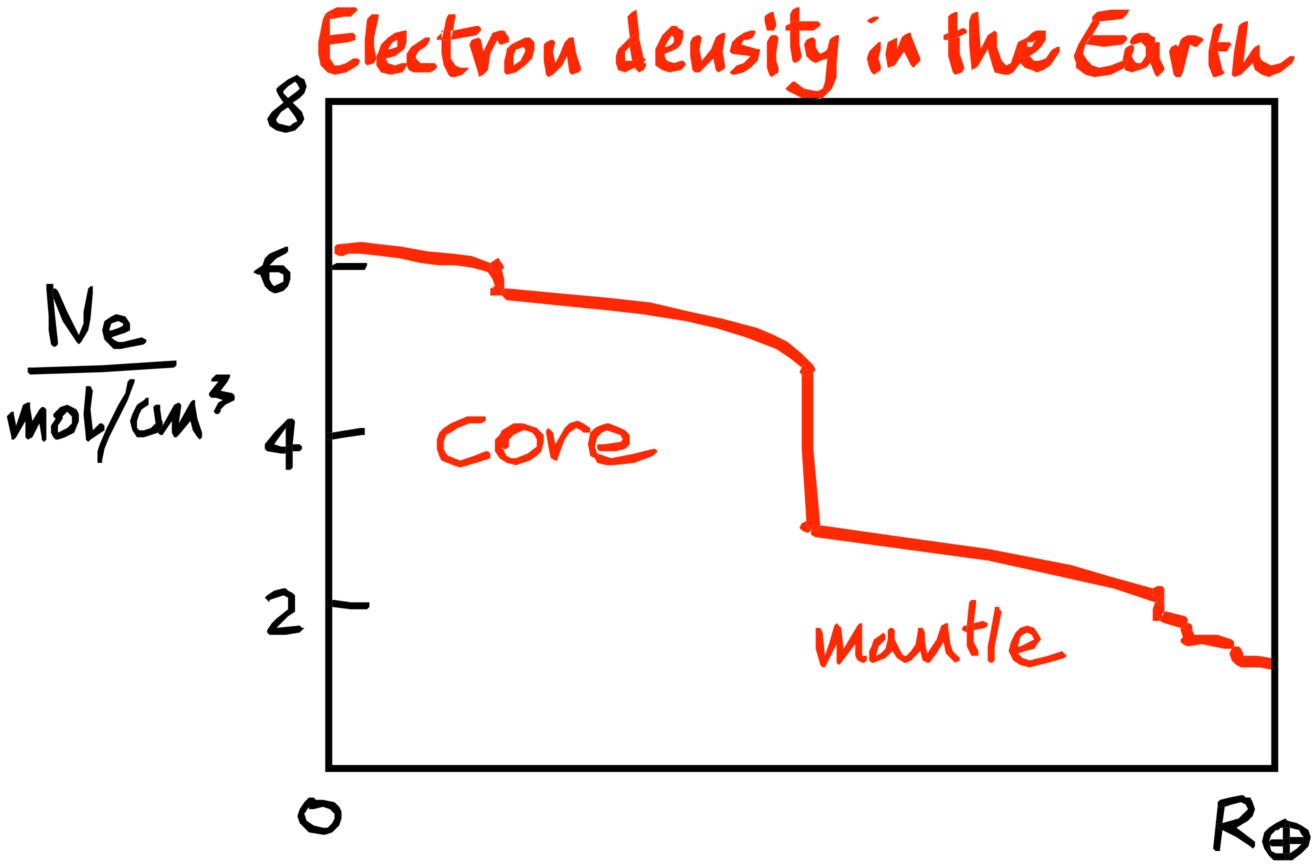
$N_e(0) \simeq 245$ mol/cm^3

$r_0 \simeq R_\odot/10.54$

But in true SSM:

$N_e(0) \simeq 100$ mol/cm^3

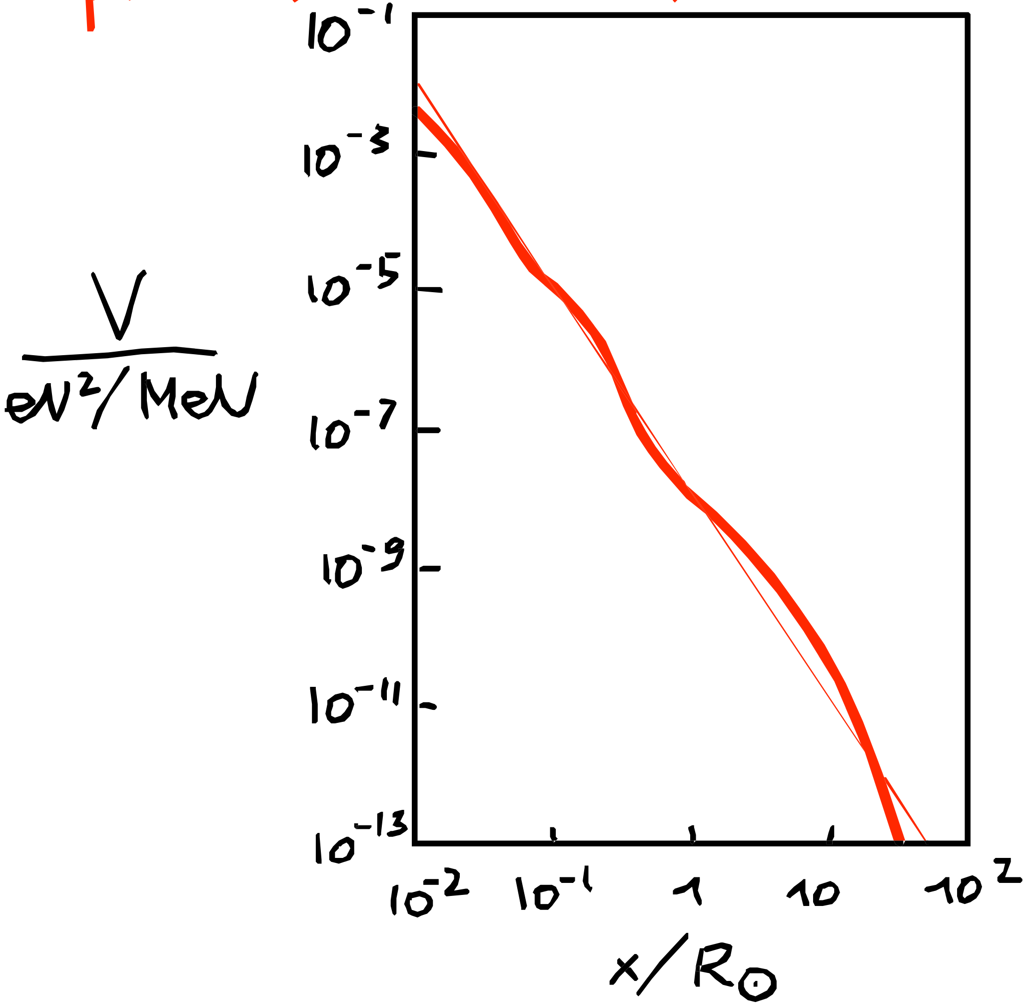
Electron density in the Earth



mantle : $N_e \sim 2 \div 3$ mol/cm^3

core : $N_e \sim 5 \div 6$ mol/cm^3

ν potential in a Supernova



$V \propto x^{-3}$

More on standard EW interaction energies

ν type	bkgd matter	Interaction energy V
ν_e	e	$\frac{1}{\sqrt{2}} G_F (4s_W^2+1)(N_e - N_{\bar{e}})$
$\nu_{\mu,\tau}$	e	$\frac{1}{\sqrt{2}} G_F (4s_W^2-1)(N_e - N_{\bar{e}})$
$\nu_{e,\mu,\tau}$	n	$\frac{1}{\sqrt{2}} G_F (N_{\bar{n}} - N_n)$
$\nu_{e,\mu,\tau}$	p	$\frac{1}{\sqrt{2}} G_F (1 - 4s_W^2)$
ν_s	e, p, n	0

for $\nu \to \bar{\nu}$:
$V \to -V$

In ordinary matter: $N_e = N_p$, $N_{\bar{e}} = N_{\bar{p}} = N_{\bar{n}} = 0$

$$V_e - V_{\mu,\tau} = \sqrt{2} G_F N_e \quad \Big\} \text{ as before}$$

$$V_\mu - V_\tau = 0 \quad \Big\} \text{ vacuum-like}$$

$$\left.\begin{aligned} V_s - V_{\mu,\tau} &= \sqrt{2} G_F \frac{N_n}{2} \\ V_s - V_e &= \sqrt{2} G_F \left(N_e - \tfrac{1}{2} N_n\right) \end{aligned}\right\} \text{relevant for sterile } \nu \text{ phenomenology}$$

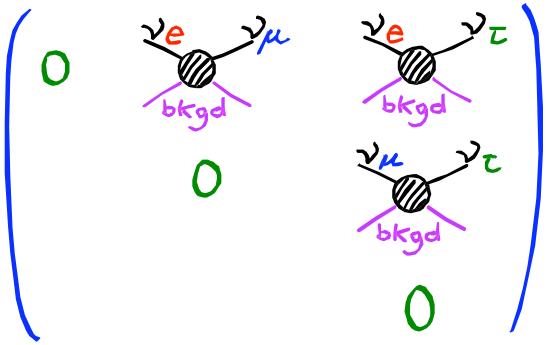
Back to 3 massless ν in matter

Standard EW inter. + ordinary matter → $H = \begin{pmatrix} p+V_{cc} & & \\ & p & \\ & & p \end{pmatrix}$

→ no off-diagonal elements in flavor basis

→ flavor is conserved

However, flavor changing neutral currents may arise in theories beyond the standard model:



$V_{FCNC} = \begin{pmatrix} 0 & \nu_e\text{–}\nu_\mu & \nu_e\text{–}\nu_\tau \\ & 0 & \nu_\mu\text{–}\nu_\tau \\ & & 0 \end{pmatrix} \propto \epsilon_{\alpha\beta} G_F N_f$

(e.g. SUSY with R-parity breaking, violations or equivalence principle ...)

In such cases, flavor transitions could take place even for massles ν

3 massive ν in vacuum, no mixing

Assume $m(\nu_\alpha) = \delta_{\alpha i}\, m_i$ $(U \equiv 1)$; then, for ultrarelativistic ν:

$$E_i = \sqrt{p^2 + m_i^2} \simeq p + \frac{m_i^2}{2p} \simeq p + \frac{m_i^2}{2E}$$

$$H = \begin{bmatrix} E_1 & & \\ & E_2 & \\ & & E_3 \end{bmatrix} \simeq \begin{bmatrix} p & & \\ & p & \\ & & p \end{bmatrix} + \frac{1}{2E} \begin{bmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{bmatrix}$$

$$= p\,\mathbb{1} + \frac{\mathcal{M}^2}{2E}$$

$$\mathcal{M}^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$$

→ H diagonal in flavor (= mass) basis

→ no flavor transitions

3 massive ν in vacuum, with mixing

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

$UU^\dagger = 1$

$m(\nu_i) = m_i$

Hamiltonian diagonal in mass basis:

$$\mathcal{H}_{mass} = \frac{\mathcal{M}^2}{2E} + p\mathbb{1}$$

$\mathcal{M}^2 = \mathrm{diag}(m_1^2, m_2^2, m_3^2)$

... but not diagonal in flavor basis

$$\mathcal{H}_{flav.} = \underbrace{U \frac{\mathcal{M}^2}{2E} U^\dagger}_{\text{off-diag}} + \underbrace{p\mathbb{1}}_{\text{diag}}$$

If no $\cancel{CP}$, U real; usual parametrization: ($\theta_{ij} \in [0, \pi/2]$)

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If $\cancel{CP}$, and mass terms are Dirac, one phase (quark-like):

$$\begin{pmatrix} c_{13} & 0 & s_{13} \\ 0 & 1 & 0 \\ -s_{13} & 0 & c_{13} \end{pmatrix} \rightarrow \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}$$

However, if $\cancel{CP}$, and mass terms are Majorana (or Dirac-Majorana):

$$U \to U U_M \quad , \quad U_M = \begin{pmatrix} 1 & & \\ & e^{i\phi'} & \\ & & e^{i\phi''} \end{pmatrix}$$

Majorana phases

... but : no effect on oscillations

$$U U_M \frac{\mathcal{M}^2}{2E} (U U_M)^\dagger = U \frac{U_M \mathcal{M}^2 U_M^\dagger}{2E} U^\dagger = U \frac{\mathcal{M}^2}{2E} U^\dagger$$

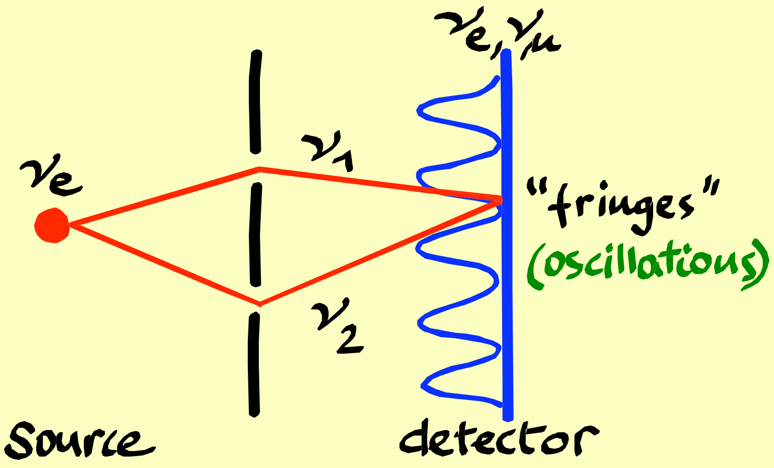
→ Oscillations do not distinguish Dirac vs Majorana neutrinos

2ν oscillations in vacuum

$\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \begin{pmatrix}c_\theta & s_\theta\\ -s_\theta & c_\theta\end{pmatrix}\begin{pmatrix}\nu_1\\ \nu_2\end{pmatrix}$; $\Delta m^2 = m_2^2 - m_1^2$

$P(\nu_e \to \nu_\mu) = \sin^2 2\theta \, \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$ see tutorials

Analogy with 2-slit expt.:



Length scales:

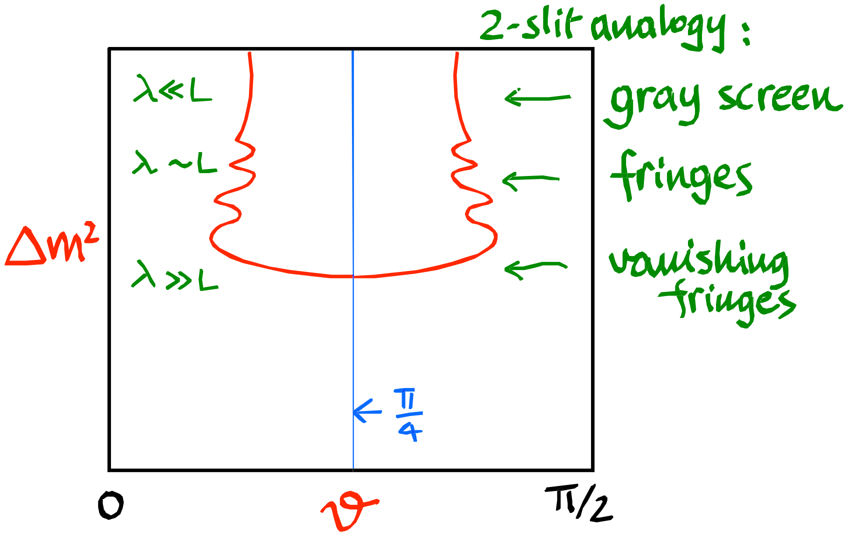
L = baseline

$\lambda = \frac{4\pi E}{\Delta m^2}$ = osc. length

Fringes may not be visible for $\lambda \ll L$ ("fast oscillations) or large expt. smearing ($\Delta E/E$ etc.)

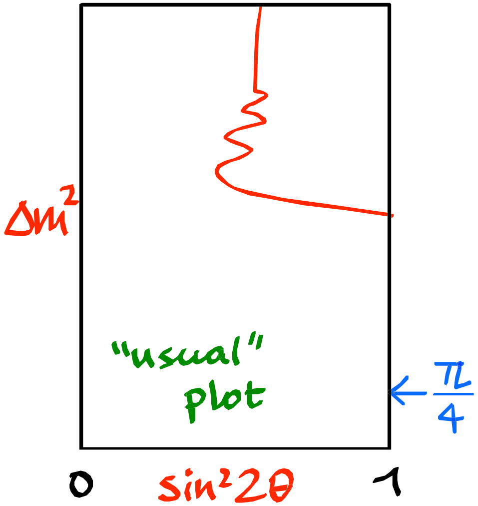
$\rightarrow \left\langle \sin^2\left(\frac{\Delta m^2 L}{4E}\right)\right\rangle \sim \frac{1}{2}$

Typical iso-$\langle P_{e\mu} \rangle$ contours



Octant symmetry: $\theta \to \frac{\pi}{2} - \theta$ in $P_{e\mu}$

If 2nd octant folded onto the 1st one:



Basically obsolete

$(\Delta m^2, \sin^2 2\theta)$ plot still used for pure 2ν $\nu_\mu \to \nu_\tau$ oscillations

(they are vacuum-like even in matter)

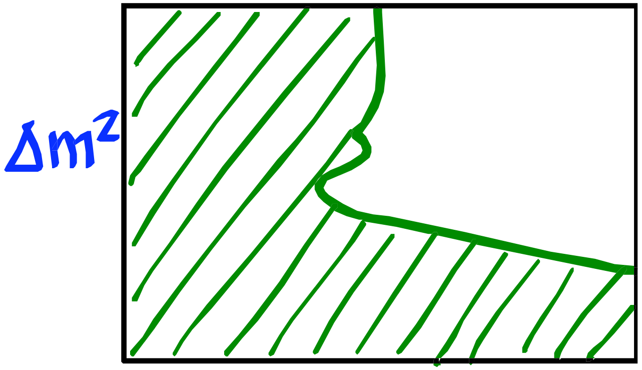
In general, better to use $\log \tan^2\theta$ or $\sin^2\theta$

(preserve octant-symmetry)

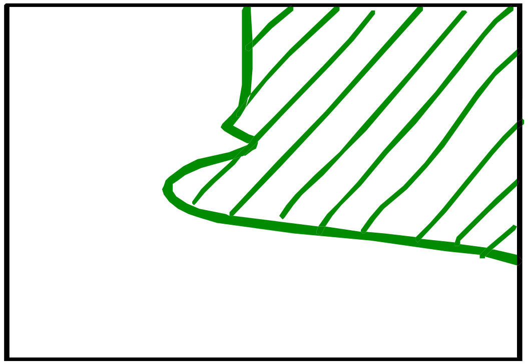
Typical experimental results

▨ allowed

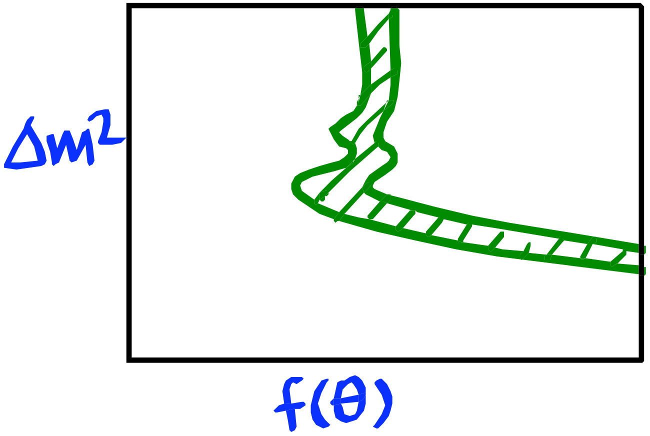
negative, $P_{\alpha\beta} <$ const



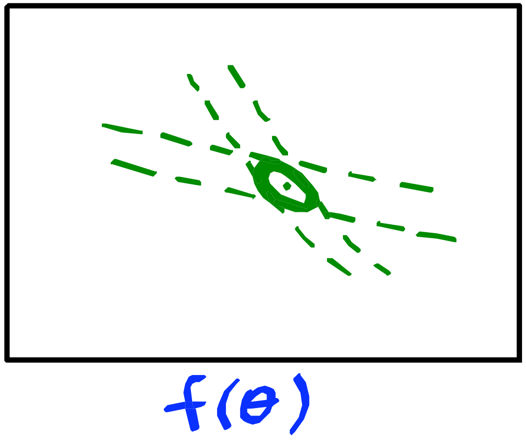
positive, $P_{\alpha\beta} > 0$



accurate, $P_{\alpha\beta} = c \pm \Delta c$



several accurate expt



2ν oscill. in constant-density matter

$$P_{e\mu} = \sin^2 2\tilde{\theta}\, \sin^2\left(\frac{\Delta\tilde{m}^2 L}{4E}\right)$$ (tutorial)

$$\sin 2\tilde{\theta} = \frac{\sin 2\theta}{\sqrt{\left(\cos 2\theta - \frac{A}{\Delta m^2}\right)^2 + \sin^2 2\theta}}$$

↑ "Breit-Wigner" resonance form

$$\frac{\Delta\tilde{m}^2}{\Delta m^2} = \frac{\sin 2\theta}{\sin 2\tilde{\theta}}$$

Can get a MSW resonant behavior for $c_{2\theta} \sim A/\Delta m^2$

→ $\Delta m^2 c_{2\theta} = 2\sqrt{2}\, G_F N_e E$

→ $\sin^2 2\tilde{\theta} \sim 1$ (enhanc.)

→ $\Delta\tilde{m}^2$ minimized

Can get suppression for $A \gg \Delta m^2 \to \sin^2 2\tilde{\theta} \sim 0$

Matter can profoundly modify osc. amplitude (enhancement - suppression) and its energy dependence. New length scale $\tilde{\lambda} = \frac{\sqrt{2}\,\pi}{G_F N_e}$ (important effects for $\lambda \sim \tilde{\lambda}$)

Note : MSW = Mikheyev-Smirnov-Wolfenstein

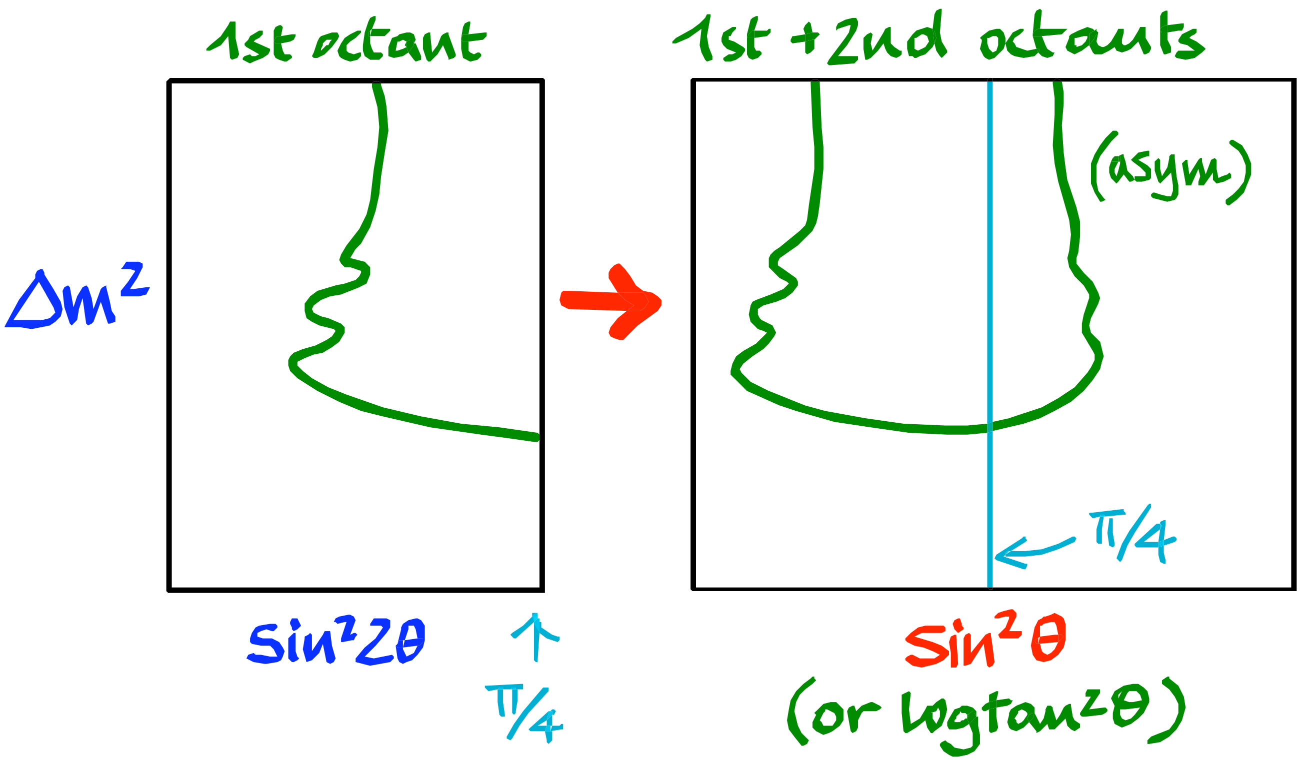
For $\bar{\nu}$: $A \to -A$ (no MSW resonance)

Matter effects are not octant-symmetric

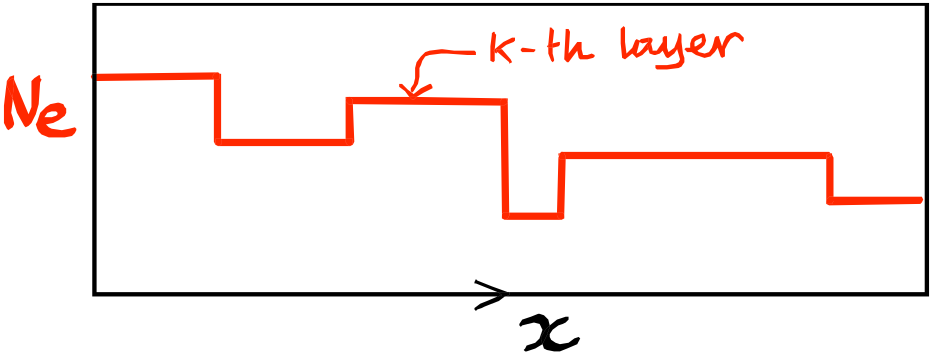
$$Q(\theta) \neq Q(\tfrac{\pi}{2}-\theta)$$

where $Q = \Delta\tilde{m}^2, \tilde{\theta}, P_{e\mu} \cdots$

→ must unfold second octant



2ν oscillations in layered matter



Approx. valid in Earth:
- Mantle + core
- Inhomogeneous crust

Tutorial: evolution operator S in 1 layer

For N layers:

$S = S_N S_{N-1} \cdot \ldots \cdot S_k \cdot \ldots \cdot S_3 S_2 S_1$

(time-ordered product)

$P_{\alpha\beta} = |S_{\alpha\beta}|^2$

can be somewhat complicated

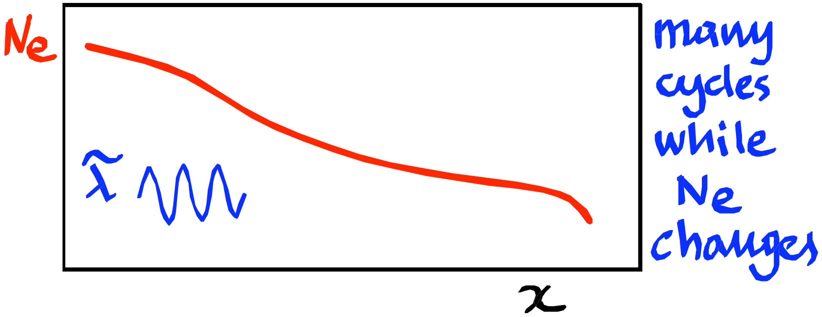
Enhancement conditions for $P_{\alpha\beta}$ contain (but do not reduce to) MSW-reson. conditions

→ Further conditions arise for constructive interference

2ν oscillations in variable density

Solution requires, in general, numerical evolution
But: Analytical approximations exist in several cases of phenomenological interest

We'll start with the case of adiabatic evolution i.e., of "slowly varying density"



We'll consider then nonadiabatic corrections to the adiabatic evolution

Note: adiabatic evolution relevant for the LMA solution to the solar ν deficit.
Nonadiabatic corrections relevant in other contexts (e.g., supernova ν)

Adiabatic evolution

At each point x :

$$\begin{pmatrix}\nu_e\\ \nu_\mu\end{pmatrix} = \begin{pmatrix}\cos\tilde\theta(x) & \sin\tilde\theta(x)\\ -\sin\tilde\theta(x) & \cos\tilde\theta(x)\end{pmatrix}\begin{pmatrix}\tilde\nu_1(x)\\ \tilde\nu_2(x)\end{pmatrix}$$

with $P(\tilde\nu_1 \to \tilde\nu_2)$
"no crossing"

Typically, $\tilde\lambda \ll L \to$ phase information lost

$\to$ can propagate "probabilities" (rather than amplitudes)

$$P(\nu_e\to\nu_e) = \underset{\substack{\uparrow\\ \text{final}\\ \nu_e}}{(1,0)}\underset{\substack{\uparrow\\ \text{rotate back}\\ \text{at } x=x_f}}{\begin{pmatrix}\cos^2\tilde\theta_f & \sin^2\tilde\theta_f\\ \sin^2\tilde\theta_f & \cos^2\tilde\theta_f\end{pmatrix}}\underset{\substack{\uparrow\\ \text{no}\\ \text{crossing}}}{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}}\underset{\substack{\uparrow\\ \text{rotate at}\\ x=x_i \text{ to}\\ \tilde\nu_{1,2}\text{ basis}}}{\begin{pmatrix}\cos^2\tilde\theta_i & \sin^2\tilde\theta_i\\ \sin^2\tilde\theta_i & \cos^2\tilde\theta_i\end{pmatrix}}\underset{\substack{\uparrow\\ \text{initial}\\ \nu_e}}{\begin{pmatrix}1\\ 0\end{pmatrix}}$$

from right to left :

$$P_{ee} = \frac{1}{2}\left(1+\cos 2\tilde\theta_i \cos 2\tilde\theta_f\right)$$

For solar neutrinos: $\tilde\theta_f = \theta$ (vacuum), up to Earth matter effects

$$P_{ee}^{\odot} = \frac{1}{2}\left(1+\cos 2\underset{\substack{\uparrow\\ \text{production point}}}{\hat\theta(x)}\cos 2\theta\right)$$

Nonadiabatic corrections

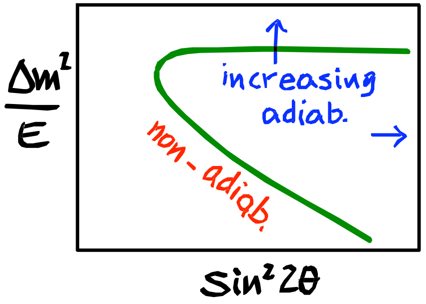
In $(\tilde{\nu}_1, \tilde{\nu}_2)$ basis: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1-P_c & P_c \\ P_c & 1-P_c \end{pmatrix}$

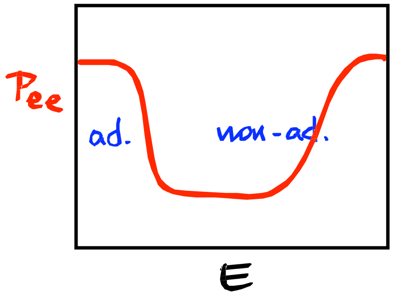
P_c = crossing prob.
$\tilde{\nu}_1 \to \tilde{\nu}_2$ "tunnelling"

$$\boxed{P_{ee} = \frac{1}{2} + \left(\frac{1}{2} - P_c\right) \cos 2\tilde{\theta}_i \cos 2\tilde{\theta}_f}$$

↑ enormous literature on P_c evaluation

Historically relevant in solar ν solutions:

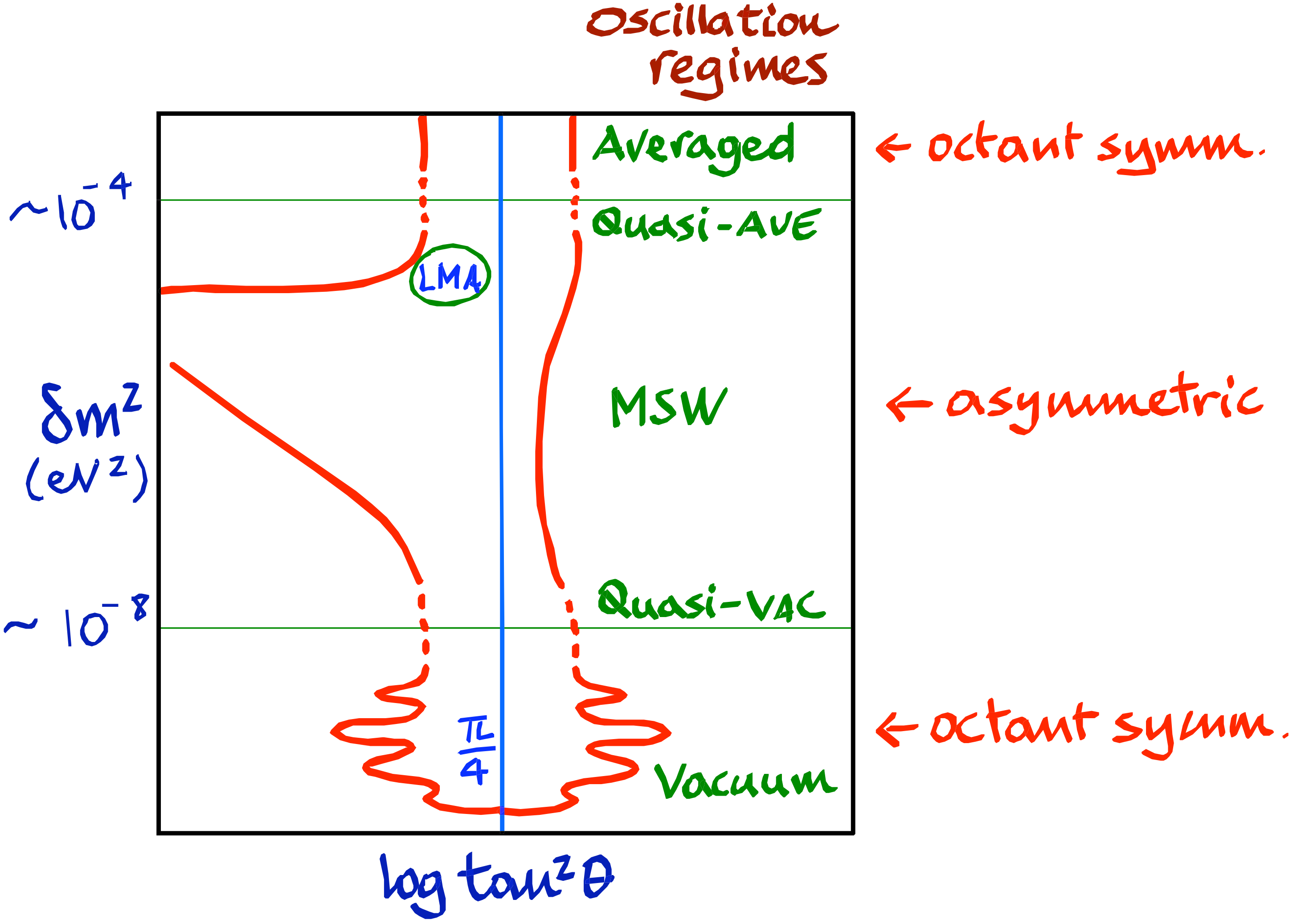




↑ MSW "triangle", zone of small P_{ee}

↑ Strong difference from vacuum case ∿∿∿

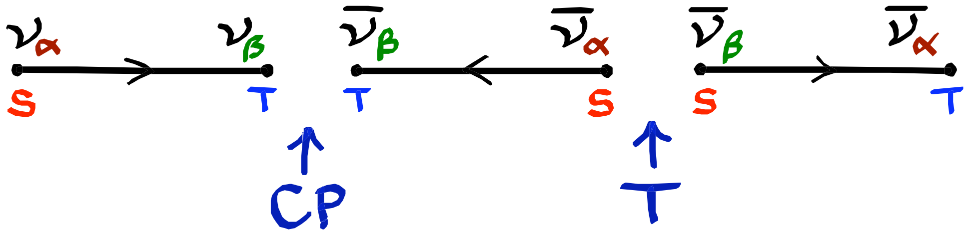
Solar P_{ee} suppression: param. space



3ν : CP violation

Requires $n \geq 3$ neutrinos;

S = source
T = target



CPT : $P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\beta \to \bar{\nu}_\alpha)$

if CP: $P(\nu_\alpha \to \nu_\beta) = P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)$

if $\cancel{CP}$: $P(\nu_\alpha \to \nu_\beta) \neq P(\bar{\nu}_\alpha \to \bar{\nu}_\beta)$

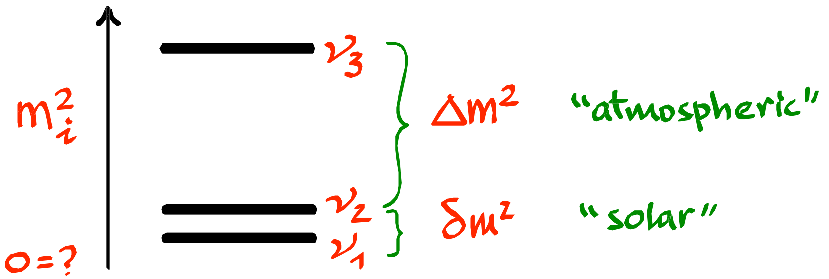
In vacuum : (See tutorials)

$$\Delta P_{\cancel{CP}} \propto \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos\theta_{13} \cdot \sin\delta \times \sin\left(\frac{\Delta m^2_{12} L}{4E}\right) \cdot \sin\left(\frac{\Delta m^2_{23} L}{4E}\right) \cdot \sin\left(\frac{\Delta m^2_{31} L}{4E}\right)$$

$$\Delta P_{\cancel{CP}} \neq 0 \text{ only if } \begin{cases} \sin\delta \neq 0 \\ \text{all } \theta_{ij} \neq 0 \\ \text{all } \Delta m^2_{ij} \neq 0 \end{cases}$$

3ν : one dominant mass scale

SQUARED MASS SPECTRUM :



(or up-side-down)

$\delta m^2 \ll \Delta m^2$

$$\mathcal{M}^2 \simeq \begin{pmatrix} 0 & & \\ & \delta m^2 & \\ & & \Delta m^2 \end{pmatrix}$$

mod $\mathbb{1}$

From the viewpoint of atmospheric ν:

$$\mathcal{M}^2 \sim \begin{pmatrix} 0 & & \\ & 0 & \\ & & \Delta m^2 \end{pmatrix}$$

→ $\not{CP}$ unobservable

From the viewpoint of solar ν:

$$\mathcal{M}^2 \sim \begin{pmatrix} 0 & & \\ & \delta m^2 & \\ & & \infty \end{pmatrix}$$

→ $\not{CP}$ unobservable

Atmospheric ν, o.d.m.s.

$\mathcal{M}^2 \sim \begin{pmatrix} 0 & & \\ & 0 & \\ & & \Delta m^2 \end{pmatrix}$, no CP $(U = U^*)$, imply in vacuum

$$P_{\alpha\alpha} = 1 - 4 U_{\alpha 3}^2 (1 - U_{\alpha 3}^2) \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

$$P_{\alpha\beta} = 4 U_{\alpha 3}^2 U_{\beta 3}^2 \sin^2\left(\frac{\Delta m^2 L}{4E}\right)$$

→ parameter space $(\Delta m^2, U_{e3}^2, U_{\mu 3}^2, U_{\tau 3}^2)$

$= (\Delta m^2, \sin^2\theta_{23}, \sin^2\theta_{13})$

Corrections to above approx. from:

- matter effects
- $\delta m^2 > 0$
- CP violation
- $\pm \Delta m^2$ (hierarchy)

Solar ν, o.d.m.s. approximation

$\mathcal{M}^2 \sim \begin{pmatrix} 0 & & \\ & \delta m^2 & \\ & & \infty \end{pmatrix}$ imply, in vacuum:

$$\begin{aligned} P_{ee} &= 1 - 4U_{e1}^2 U_{e2}^2 \sin^2\left(\frac{\delta m^2 L}{4E}\right) \\ &\quad - 4U_{e1}^2 U_{e3}^2 \sin^2(\infty) \\ &\quad - 4U_{e2}^2 U_{e3}^2 \sin^2(\infty) \end{aligned}$$

$\leftarrow \sin^2(\infty) \sim \frac{1}{2}$

$$= (1-U_{e3}^2)^2 - 4U_{e1}^2 U_{e2}^2 \sin^2\left(\frac{\delta m^2 L}{4E}\right) + U_{e3}^4$$

$$= \cos^4\theta_{13}\left[1 - \sin^2 2\theta_{12} \sin^2\left(\frac{\delta m^2 L}{4E}\right)\right] + \sin^4\theta_{13}$$

$\rightarrow$ parameter space $(\delta m^2, U_{e1}^2, U_{e2}^2, U_{e3}^2)$

$= (\delta m^2, \sin^2\theta_{12}, \sin^2\theta_{13})$

Structure $P_{3\nu} = c_{13}^4 P_{2\nu} + s_{13}^4$
preserved in matter (with $V \rightarrow c_{13}^2 V$)
Parameter space unaltered,
negligible corrections from $\Delta m^2 < \infty$

3ν: more precise definitions and conventions

Previous notation somewhat "sloppy":

- ν = field (QF operator),
  state (QM ket),
  component (number) ?
- U, U^* ?
- $\Delta m^2 = m_3^2 - m_1^2$ or $m_3^2 - m_1^2$
- $\Delta m^2 \gtrless 0$, and δm^2 ?
- CP phases ?

But: Important to be precise, e.g., in prospective NuFact studies

Consistent use of U & U*

Fields: $\nu_{\alpha L} = \sum_{i=1}^{3} U_{\alpha i}\,\nu_{iL}$

(quantized, in the CC lagrangian)

particle created by $\psi^{\dagger}|0\rangle$ or $\overline{\psi}|0\rangle$

↓

States: $|\nu_\alpha\rangle = \sum_i U^*_{\alpha i}\,|\nu_i\rangle$

(one-particle kets)

↓

Components: if $|\nu\rangle = \sum_i \nu^i |\nu_i\rangle$

$= \sum_\alpha \nu^\alpha |\nu_\alpha\rangle$

then

$$\nu^\alpha = \sum_i U_{\alpha i}\,\nu^i$$

(components = numbers)

e.g. $|\nu_e\rangle$ components: $\nu^e = \begin{pmatrix}1\\0\\0\end{pmatrix}$

in flavor basis

$$U = O_{23}\,\Gamma_\delta\, O_{13}\,\Gamma_\delta^{+}\, O_{12}$$

$$O_{ij} \ni \begin{pmatrix} c_{ij} & s_{ij} \\ -s_{ij} & c_{ij} \end{pmatrix} \text{ at } (ij) \;;\quad \Gamma_\delta = \mathrm{diag}(1,1,e^{i\delta})$$

Note: no $\cancel{CP}$ → $U = U^*$

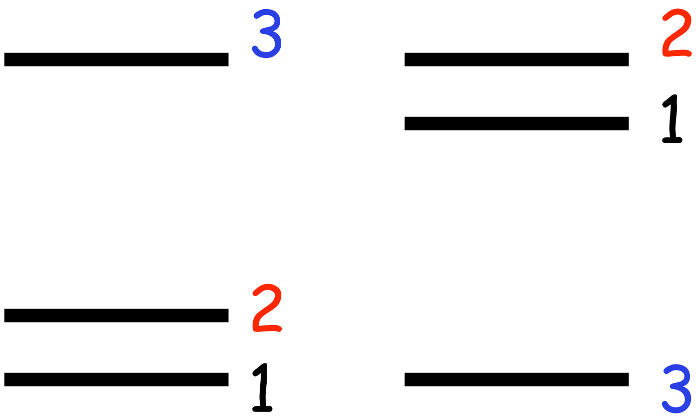
Satisfied for $\delta = 0$ AND $\delta = \pi$: $\Gamma_\delta = (1, 1, \pm 1)$

$$\left.\begin{aligned} \delta = 0 :&\quad \Gamma_\delta\, O_{13}\,\Gamma_\delta^{+} = O_{13} \\ \delta = \pi :&\quad \Gamma_\delta\, O_{13}\,\Gamma_\delta^{+} = O_{13}\Big|_{s_{13}\to -s_{13}} \end{aligned}\right\}\quad \begin{aligned} \cos\delta &= +1 \\ \cos\delta &= -1 \end{aligned}$$

→ Two CP-conserving cases

Masses: labels and splittings

Consensus labels: doublet=(ν_1, ν_2), with ν_2 heaviest in both hierarchies



$$\delta m^2 = m_2^2 - m_1^2 > 0$$

Sign of smallest splitting: conventional. The relative ν_e content of ν_1 and ν_2 is instead physical (given by MSW effect)

$$\text{Note}: \quad |m_3^2 - m_1^2| = \begin{cases} \text{largest splitting (N.H.)} \\ \text{next−to−largest splitting (I.H.)} \end{cases}$$

$$\Rightarrow \Delta m_{31}^2 \text{ (or } \Delta m_{32}^2\text{) change physical meaning from NH to IH}$$

We prefer to define the 2nd independent splitting as:

$$\Delta m^2 = \left| \frac{\Delta m^2_{31} + \Delta m^2_{32}}{2} \right| = \left| m^2_3 - \frac{m^2_1 + m^2_2}{2} \right|$$

so that the largest and next-to-largest splittings, in both NH & IH, are given by: $\Delta m^2 \pm \frac{\delta m^2}{2}$

and only one physical sign distinguishes NH (+) from IH (-), as it should be:

$$(m^2_1, m^2_2, m^2_3) = \frac{m^2_2 + m^2_1}{2} + \left(-\frac{\delta m^2}{2}, +\frac{\delta m^2}{2}, \pm \Delta m^2 \right)$$

sign($\pm\Delta m^2$) can be determined - in principle - by interference of Δm^2-driven oscillations with some Q-driven oscillations, provided that sign(Q) is known. Two ways (barring exotics):

Q = A(x) = $2\sqrt{2}G_F N_e(x)E$ **(only in matter & for s_{13}>0)**

Q = δm^2 **(in any case, but hard !)**

Weak sensitivity with current data; challenge for future expts.

Majorana phases

$$U \to U \cdot U_M$$

Useful (not unique) convention:

$$U_M = \mathrm{diag}\left(1, e^{\frac{i}{2}\phi_2}, e^{\frac{i}{2}(\phi_3+2\delta)}\right)$$

includes U_M

$$\to m_{\beta\beta} = \left|\sum_i U_{ei}^2 m_i\right|$$

$$= \left|c_{13}^2 c_{12}^2 m_1 + c_{13}^2 s_{12}^2 m_2 e^{i\phi_2} + s_{13}^2 m_3 e^{i\phi_3}\right|$$

↑ does not contain δ explicitly

Besides $m_{\beta\beta}$, two relevant observables sensitive to absolute ν masses:

$$m_\beta = \left[\sum_i |U_{ei}^2| m_i^2\right]^{1/2}$$

$$= \left[c_{13}^2 c_{12}^2 m_1^2 + c_{13}^2 s_{12}^2 m_2^2 + s_{13}^2 m_3^2\right]^{1/2}$$

(β-decay)

$$\Sigma = m_1 + m_2 + m_3$$

(cosmology)

In the next lecture, we shall see how
the mass and mixing parameters are probed
by oscillation and non-oscillation experiments