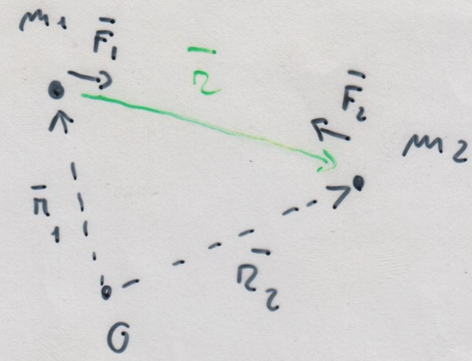
# ELEMENTI ORBITALI

- Moto relativo

- Equazione di Keplero

- Trasformazione da elementi orbitali a coordinate cartesiane e viceversa.

# EQUAZIONE del MOTO RELATIVO.




$\bar{r}_1 + \bar{r} = \bar{r}_2$

$\bar{r} = \bar{r}_2 - \bar{r}_1$

$$\bar{F}_1 = G\frac{m_1 m_2}{r^3}\,\bar{r} = m_1 \ddot{\bar{r}}_1$$ Forza su $m_1$

$$\bar{F}_2 = -G\frac{m_1 m_2}{r^3}\,\bar{r} = m_2 \ddot{\bar{r}}_2$$ Forza su $m_2$

$$m_1 \ddot{\bar{r}}_1 + m_2 \ddot{\bar{r}}_2 = 0 \Rightarrow \bar{a}_{CM}$$ (No forze esterne)

$$\ddot{\bar{r}} = \ddot{\bar{r}}_2 - \ddot{\bar{r}}_1 = \left(-\frac{G m_1 m_2}{r^3 m_2} - \frac{G m_1 m_2}{r^3 m_1}\right)\bar{r} =$$

$$= -G\frac{(m_1 + m_2)}{r^3}\,\bar{r}$$

Eq. in $\bar{r}$

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# ELEMENTI ORBITALI: DEFINIZIONE

EQUAZIONE del MOTO RELATIVO: $\frac{d^2\bar{r}}{dt^2} + \mu \frac{\bar{r}}{r^3} = 0$

$\mu = G(m_1 + m_2)$

- SISTEMA È ISOLATO → 3 - 1 GRADI di LIBERTÀ → IL MOTO SI SVOLGE SU UN PIANO.

INTEGRALE MOMENTO ANGOLARE: $\bar{h} = \bar{r} \times \dot{\bar{r}}$

IN COORDINATE POLARI:

- $\ddot{\bar{r}} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + \left[\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right]\hat{\theta}$

$\ddot{\bar{r}} + \mu \frac{\bar{r}}{r^3} = 0 \;\to\; \ddot{r} - r\dot{\theta}^2 + \mu \frac{r}{r^3} = 0 \quad \hat{r}$

$\phantom{\ddot{\bar{r}} + \mu \frac{\bar{r}}{r^3} = 0} \;\to\; \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \quad \hat{\theta}$

- $\bar{h} = r^2\dot{\theta}\,\bar{z} =$ COSTANTE

$\left(\bar{h} = r\hat{r} \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = \right.$

$= 0 + r^2\dot{\theta}\,\hat{z}$

2 -

$$\left(\ddot{r} - r\dot{\theta}^2\right) + \frac{\mu}{r^2} = 0$$ → cerchiamo $r(\theta)$

variabile ausiliaria $y = \frac{1}{r}$

$$\dot{r} = -h\frac{dy}{d\theta} \rightarrow \ddot{r} = -h^2 y^2 \frac{d^2y}{d\theta^2} \rightarrow$$

$$-h^2y^2\frac{d^2y}{d\theta^2} - h^2y^3 + \mu y^2 = 0 \Rightarrow$$

$$\frac{d^2y}{d\theta^2} + y = \frac{\mu}{h^2}$$ eq. differenziale a coefficienti costanti

$$y(\theta) = \frac{\mu}{h^2}\left(1 + e\cos(\theta - \theta_0)\right)$$

$$\boxed{r(\theta) = \frac{p}{1 + e\cos(\theta - \theta_0)}}$$

$$p = \frac{h^2}{\mu}$$

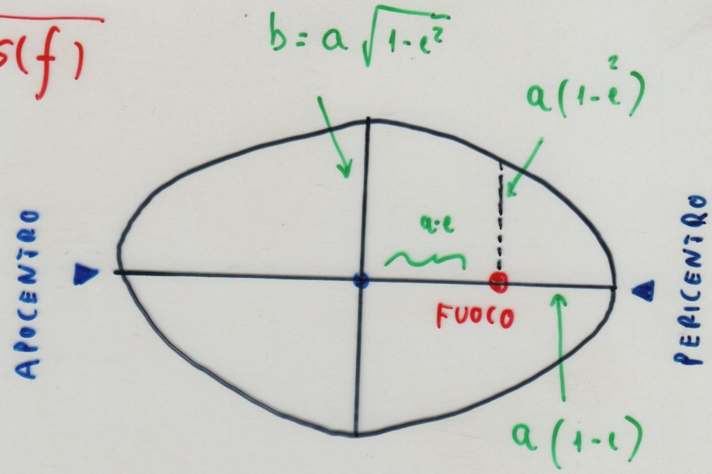
ELLISSI: $p = a(1 - e^2)$ $e < 1$

PARABOLA: $p = 2q$ $e = 1$ con $q$ distanza minima

IPERBOLE: $p = a(e^2 - 1)$ $e > 1$

$\boxed{f = \text{ANOMALIA VERA}}$

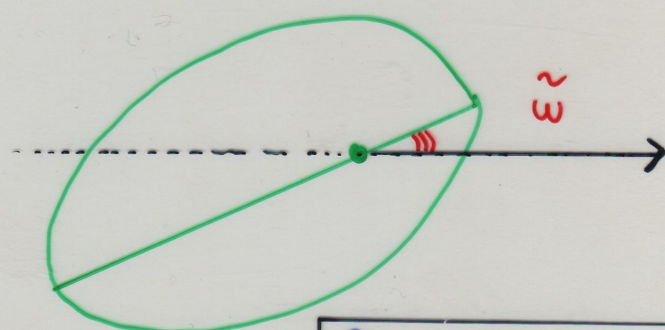
$$r = \frac{a\,(1-e^2)}{1 + e\cos(f)}$$




$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\begin{cases} x = a \cos u \\ y = b \sin u \end{cases}$$

• DIREZIONE di RIFERIMENTO:




$\boxed{\tilde{\omega} = \text{LONGITUDINE del PERICENTRO}}$

AREA $= \pi a b$

PERIMETRO $: 4\pi\, E(e)$

$$E(e) = \int_0^{\frac{\pi}{2}} \sqrt{1 - e^2 \sin^2\theta}\; d\theta$$

Integrale ellittico completo del 2° tipo.

$V_{A_2} = \frac{h}{2} \Rightarrow \frac{hT}{2} = \pi ab = \pi a^2 \sqrt{1-e^2}$

$h = \sqrt{\mu a(1-e^2)} \rightarrow \frac{h^2}{\mu} = a(1-e^2)$

$$\boxed{\begin{array}{ll} T = 2\pi \sqrt{\dfrac{a^3}{\mu}} & \text{PERIODO} \\ n = \dfrac{2\pi}{T} = \sqrt{\dfrac{\mu}{a^3}} & \text{MOTO MEDIO} \end{array}}$$

COSTANTE del MOTO n° 2: INTEGRALE ENERGIA

$$\frac{1}{2}v^2 - \frac{\mu}{r} = C$$

$\dot{\bar{r}} \cdot \ddot{\bar{r}} + \mu \frac{\dot{\bar{r}} \cdot \bar{r}}{r^3} = 0$

$= \frac{d}{dt}\left(\frac{\dot{\bar{r}} \cdot \dot{\bar{r}}}{2} - \frac{\mu}{r}\right)$

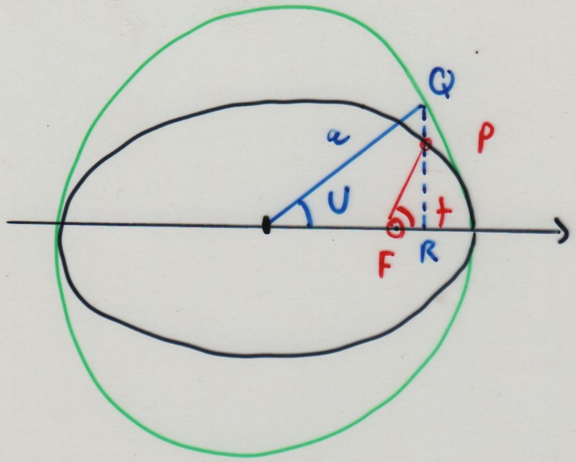
$h =$ (INVARIANTE
INTEGRALE MOMENTO ANGOLARE

$C =$ (INVARIANTE
INTEGRALE ENERGIA

- IL SISTEMA HA 3 - 2 GRADI di LIBERTÀ $\Rightarrow$ **1**

$$\boxed{M = n(t - t_0) \quad \text{ANOMALIA MEDIA}}$$

# EQUAZIONE di KEPLERO




1) $r(U)$
$f(U)$

2) Eq. che dà $U(t)$

• 1° OBIETTIVO: Calcolare $r, f$ in funzione di $U$

$$FR = r\cos f = a\cos U - ae$$
$$PR = r\sin f = \sqrt{1-e^2}\, a\sin U$$

$$\boxed{\begin{aligned} r &= a(1-e\cos U) \\ \operatorname{tg}\left(\frac{f}{2}\right) &= \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \operatorname{tg}\left(\frac{U}{2}\right) \end{aligned}}$$

* $\cos f = 1 - 2\sin^2\frac{f}{2}$ $\qquad r\cos f = r\left(1-2\sin^2\frac{f}{2}\right)$

$$2r\sin^2\frac{f}{2} = r - r\cos f = a(1-e\cos U) - a\cos U + ae = a(1+e)(1-\cos U)$$

$\cos f = 2\cos^2\frac{f}{2} - 1 \Rightarrow 2r\cos^2\frac{f}{2} = a(1-e)(1+\cos U)$

• OBIETTIVO: CALCOLARE $U(t)$

$\dot{r} = \frac{na}{\sqrt{1-e^2}}\, e \sin f \qquad r\dot{f} = \frac{na}{\sqrt{1-e^2}}(1+e\cos f)$

1) $$\mu\left(\frac{2}{r} - \frac{1}{a}\right) = V^2 = \dot{r}^2 + (r\dot{f})^2$$

$$\dot{r} = \frac{na\, e \sin f}{\sqrt{1-e^2}} \qquad (M\text{-}D,\ pg\ 31) \qquad C = -\frac{\mu}{2a}$$

$$r\dot{f} = \frac{na(1+e\cos f)}{\sqrt{1-e^2}}$$

2) $$\begin{cases} \dot{r} = \frac{na}{r}\sqrt{a^2e^2 - (r-a)^2} = \frac{na^2 e \sin U}{a(1-e\cos U)} \\ \dot{r} = ae \sin U \frac{dU}{dt} \end{cases} \Rightarrow \frac{dU}{dt} = \frac{n}{(1-e\cos U)}$$

3) $$\boxed{U - e\sin U = M = n(t-T_0)}$$

Equazione di Keplero $\Rightarrow$ Legge Oraria

SOLUZIONE: Metodo iterativo

$$U_0 = M$$

$$U_1 = U_0 + e\sin U_0$$

$$U_2 = U_0 + e\sin U_1$$

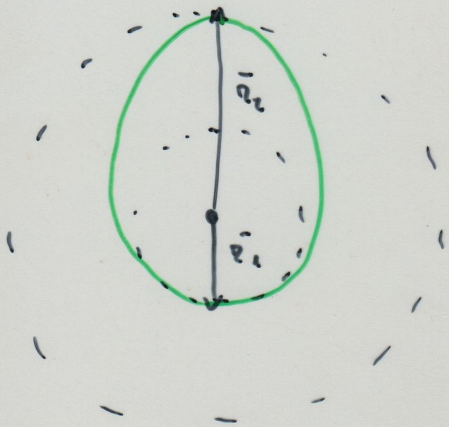
$$\vdots$$

$$U_{i+1} = M + e\sin U_i$$

$$\vdots$$

1-0

# TRASFERIMENTO di Hohmann




ORBITA di TRASFERIMENTO:

$$a(1-e) = r_1$$

$$a(1+e) = r_2$$

$$a(1-e) + a(1+e) = 2a = r_1 + r_2$$

$$a = \frac{r_1 + r_2}{2} \qquad e = 1 - \frac{r_1}{a}$$

Calcolo del $\Delta V$:

$$V_{C1} = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{\frac{\mu}{r_1}} \qquad V_{C2} = \sqrt{\frac{\mu}{r_2}}$$

$$V_{P1} = \sqrt{\mu\left(\frac{2}{a(1-e)} - \frac{1}{a}\right)} = \sqrt{\frac{\mu}{a}\left(\frac{1+e}{1-e}\right)} \qquad V_{A2} = \sqrt{\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)}$$

$$\Delta V_1 = V_{P1} - V_{C1} = \sqrt{\frac{\mu}{a}\left(\frac{1+e}{1-e}\right)} - \sqrt{\frac{\mu}{r_1}}$$

$$\Delta V_2 = V_{C2} - V_{A2} = \sqrt{\frac{\mu}{r_2}} - \sqrt{\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)}$$

TEMPO di VOLO: mezzo PERIODO $T_{oj} = \pi\sqrt{\frac{a^3}{\mu}}$

## ESEMPI:

Satellite con $e=0$ e $r = 1.2769 \times 10^4$ Km.

Calcolare $\Delta V$ minima per raddoppiare altitudine.

$r_1 = 1.2769 \times 10^4$ Km $(= 2r_\oplus)$

$r_2 = 3r_\oplus = 1.9154 \times 10^4$ Km

$a = \frac{5}{2} r_\oplus = 1.5961 \times 10^4$ Km $\quad e = 0.2$

$V_{c1} = \sqrt{\frac{\mu}{r_1}} = 5.595 \times 10^3$ m/s $\quad V_{P1} = 6.129 \times 10^3 \frac{m}{s}$

$\Delta V_1 = 0.534 \times 10^3$ m/s

$V_{c2} = 4.569 \times 10^3$ m/s $\quad V_{A2} = 4.086 \times 10^3 \frac{m}{s}$

$\Delta V_2 = 0.482 \times 10^3$ m/s

- $\Delta V_{tot} = 1.016 \times 10^3 \frac{m}{s}$

## TRASFERIMENTO di Hohmann su MARTE

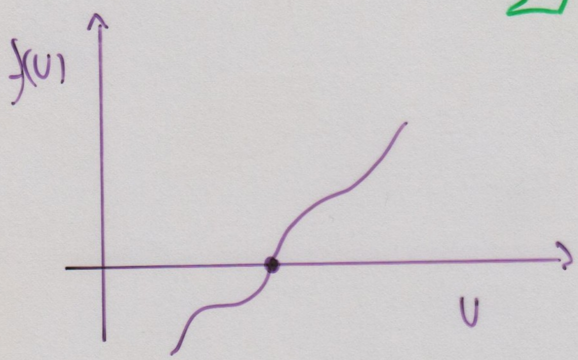
$a = \frac{1+1.5}{2} = 1.25$ AU $\quad e = 0.2$

$T_{OF} = \frac{T_{ab}}{2} = 255$ giorni $\sim 8.5$ mesi

# ALGORITMO NUMERICO per la RAPIDA CONVERGENZA di Eq. KEPLERO

$$\begin{cases} f(U) = U - e \sin U - M = 0 \\ f'(U) = 1 - e \cos U > 0 \quad \text{per } e \leq 1 \end{cases}$$

$\Rightarrow$ SOLUZIONE UNICA ☺




METODO di Newton-Raphson (vg. 256 Numerical Recipes)

- $X_0 = x \quad (= U)$

$$f(x+\delta) = f(x) + f'(x) \cdot \delta + \dots = 0$$

- $X_1 = x + \delta \quad \text{con } \delta = -\dfrac{f(x)}{f'(x)}$

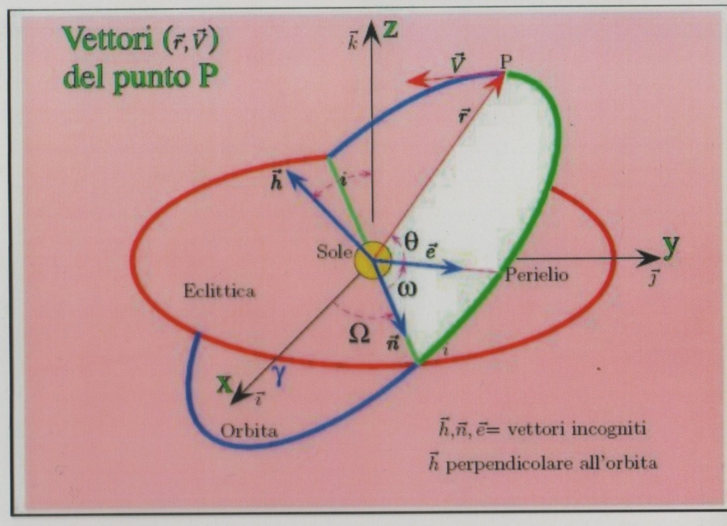
$\vdots$

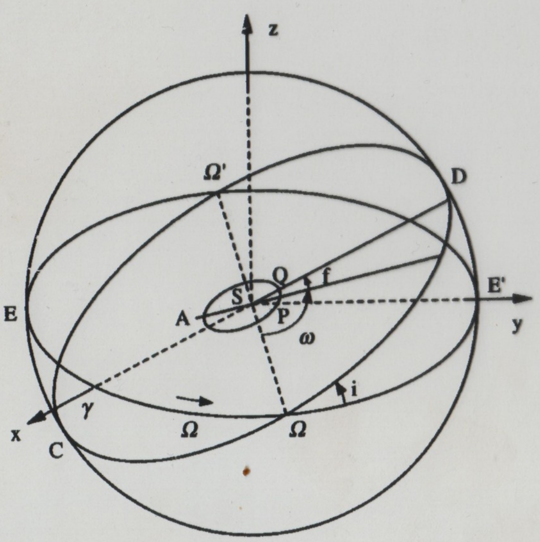
- $x_{i+1} = x_i + \delta = x_i - \dfrac{f(x_i)}{f'(x_i)}$

CONVERGENZA QUADRATICA! molto rapida.

(vg 35 D-M, metodo DANBY, convergenza quartica!)

## *Orbital elements*




**A semimajor axis,**
**e eccentricity,**
**i inclination,**
**M mean anomaly,**
**ω pericenter argument,**
**Ω node longitude**

# TRASFORMAZIONI

CASO PIANO

$a, e, M \longleftrightarrow \bar{x}, \bar{v}$

1) da $M$ a $U$ $\quad U - e \sin U = M$

2) $r = a(1 - e \cos U)$

$\text{tg}\left(\frac{f}{2}\right) = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \text{tg}\left(\frac{U}{2}\right) \Rightarrow r, f$

3) $x = r \cos f \qquad y = r \sin f$

$x = a(\cos U - e) \qquad y = a\sqrt{1-e^2} \sin U$

$\left(\frac{dU}{dt} = \frac{n}{1-e\cos U}\right)$

4) $V_x = -\frac{na \sin U}{1 - e\cos U} \qquad V_y = \frac{na\sqrt{1-e^2}\cos U}{1 - e\cos U}$

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-1) $r = \sqrt{x^2+y^2+z^2} \qquad v = \sqrt{V_x^2+V_y^2+V_z^2}$

-2) $C = \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a}$

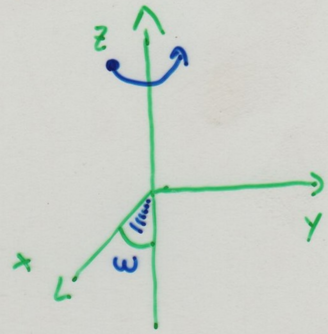
-3) $h = |\bar{r} \times \bar{v}| = \sqrt{\mu a(1-e^2)}$

-4) $r = \frac{a(1-e^2)}{1+e\cos f} \qquad \dot{r} = \frac{nae \sin f}{\sqrt{1-e^2}} = \frac{\bar{v}\cdot\bar{r}}{r}$

-5) $\text{tg}\left(\frac{f}{2}\right) = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \text{tg}\left(\frac{U}{2}\right) \qquad U - e\sin U = M$

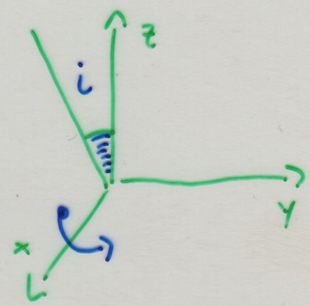
# ORBITA NELLO SPAZIO 3D

$$M_1 = \begin{pmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Attorno asse Z



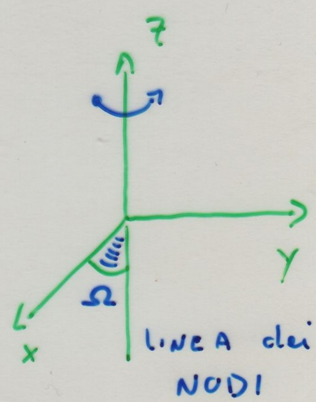

$\omega$ = argomento pericentro

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix}$$ Attorno asse X




$i$ = inclinazione

$$M_3 = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Attorno asse Z




$\Omega$ = longitudine nodo

$$\tilde{\omega} = \omega + \Omega$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = M_3 M_2 M_1 \begin{pmatrix} r\cos f \\ r \sin f \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} r\cos\Omega\cos(\omega+f) - r\sin\Omega\sin(\omega+f)\cos i \\ r\sin\Omega\cos(\omega+f) + r\cos\Omega\sin(\omega+f)\cos i \\ r\sin(\omega+f)\sin i \end{pmatrix}$$

ps 53
D-M

11-b

# TRASFORMAZIONE da ELEMENTI KEPLERIANI a COORDINATE CARTESIANE.

$a, e, i, \omega, \Omega, M \Rightarrow \bar{x}, \bar{v}$

1) Eq. Keplera $U - e \sin U = M \to f$

2) $x = a(\cos U - e)$

$y = a\sqrt{1-e^2}\sin U$

3) $V_x = -\dfrac{n a \sin U}{1 - e\cos U}$

$V_y = \dfrac{n a \sqrt{1-e^2}\cos U}{1 - e\cos U}$

$a, e, M \to (x, y, 0)\ (V_x, V_y, 0)$

4)

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = M_3(\Omega)\cdot M_1(i)\cdot M_3(\omega)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} V_X \\ V_Y \\ V_Z \end{pmatrix} = \cdots\cdots \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} V_x \\ V_y \\ 0 \end{pmatrix}$$

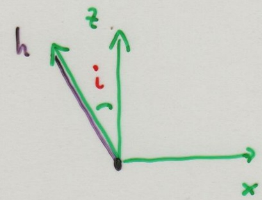
10-b

SAPENDO $x, y, z$ TROVARE GLI ANGOLI ORBITALI
$V_x, V_y, V_z$

52-53
D-M

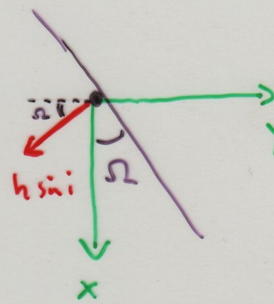
- $i$

$$h \cos i = h_z$$




- $\Omega$

$$\begin{cases} h \sin i \sin\Omega = \pm h_x \\ h \sin i \cos\Omega = \mp h_y \end{cases}$$




$$\operatorname{tg}\Omega = -\left(\frac{h_x}{h_y}\right)$$

- $\omega$

$$\begin{cases} \sin(\omega + f) = \dfrac{z}{r \sin i} \\ \cos(\omega + f) = \dfrac{x}{r \cos\Omega} + \dfrac{z}{r}\,\dfrac{\sin\Omega}{\cos\Omega}\,\dfrac{\cos i}{\sin i} \end{cases}$$

$$\Downarrow$$

$$\omega + f \Rightarrow \omega$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = M_3 M_2 M_1 \begin{pmatrix} r \cos f \\ r \sin f \\ 0 \end{pmatrix}$$

# TRASFORMAZIONE da COORDINATE CARTESIANE $\Rightarrow$ ELEMENTI KEPLERIANI

$$\dot{r} = \frac{\bar{v} \cdot \bar{r}}{r}$$

$$\bar{h} = \bar{r} \times \bar{v} = h\,\hat{h}$$

1) $r = \sqrt{x^2+y^2+z^2}$ $\quad V = \sqrt{V_x^2+V_y^2+V_z^2}$

$$C = \frac{1}{2}V^2 - \frac{\bar{\mu}}{r} \qquad C = -\frac{\bar{\mu}}{2a} \qquad h^2 = \bar{\mu}\, a(1-e^2) \quad \Rightarrow \quad \boxed{a, e}$$

2) $h \cos i = h_z$

$$\boxed{i = \arccos\left(\frac{h_z}{h}\right)}$$

3) $h \sin i \sin\Omega = \pm h_x$

$h \sin i \cos\Omega = \mp h_y$

$$\boxed{tg(\Omega) = -\left(\frac{h_x}{h_y}\right)}$$

4) $\cos f = \dfrac{a(1-e^2)}{r \cdot e} - \dfrac{1}{e}$

$\sin f = \dfrac{\dot{r}\sqrt{1-e^2}}{nae}$

$$\boxed{tg(f) = \frac{\dot{r}\, r\sqrt{1-e^2}}{na(a(1-e^2)-r)}}$$

5) $\sin(\omega+f) = \dfrac{z}{r \sin i}$ $\quad \Rightarrow \omega + f \Rightarrow \boxed{\omega}$

$$\cos(\omega+f) = \frac{x}{r\cos\Omega} + \frac{z}{r}\,\frac{\sin\Omega}{\cos\Omega}\,\frac{\cos i}{\sin i}$$

6) $tg\left(\dfrac{U}{2}\right) = \left(\dfrac{1-e}{1+e}\right)^{\frac{1}{2}} tg\left(\dfrac{f}{2}\right) \Rightarrow U \Rightarrow$

$$M = U - e\sin U \quad \Rightarrow \boxed{M}$$

10) **PROBLEMA 4**: Calcolare l'intervallo tra 2 congiunzioni di Marte e Terra e mostrare che la distanza minima alla congiunzione può variare di un fattore 2.

$G = 6.6743 \cdot 10^{-11} \frac{m^3}{kg\, s^2}$

$K = 0.01720 21$

$GM_{\odot} = K^2 \frac{AU^3}{day^2} \rightarrow 2.9591 \times 10^{-4}$

$a_M = 1.5236631$ AU

$a_T \simeq 1.000..$

$m_M = 9.146 \times 10^{-3} \frac{1}{day}$ $T_M = 686.96$ day

$m_T = 1.720 \times 10^{-2} \frac{1}{day}$ $T_T = 365.25$ day

$\frac{T_M}{T_T} \simeq 1.88$ circa 2

$m_M \Delta t = m_E \Delta t - 2\pi \Rightarrow \Delta t = \frac{2\pi}{m_E - m_M} \simeq 780$ day

af. pr.

$D_{min} = -a_T(1+e_T) + a_M(1-e_M) = 0.0365$ AU

$D_{MAX} = -a_T(1-e_T) + a_M(1+e_M) = 0.0683$ AU

pr. af.