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Chapter 1

Stochastic processes

In practical terms, a stochastic process is a time-evolution phenomenon whose analysis requires probability theory. From an empirical point of view, a stochastic process is characterized by the set of the different realizations of the process itself, where a single realization is obtained by recording a number of outcomes during the experiment elapsing time. The random nature of the time-evolution implies that a different realization of the same experiment produces a completely different pattern for the observed physical quantities.

Some examples of stochastic processes are:

- The binomial process, consisting e.g. in recording the "head" and "tail" sequence during a series of coin tosses;
- The Brownian process, consisting in recording the position of a mesoscopic particle in a fluid suspension;
- The Poisson process, consisting e.g. in counting the number of people randomly arriving and leaving a queue;
- The epidemic spread process, defined by the number of individual infected by a certain disease;
- Meteorological processes, e.g. the number of hours during a day in which the sun shines on a certain location;
- A finance process, where the value of a stock price is detected for various days.

The random nature of the processes is due in general to the presence of a large number of "agents" whose unpredictable different actions determine the overall evolution of the process. For example, a Brownian particle is displaced as the result of the collisions with the fluid particles. The dynamical laws describing these collisions (both at the classical and at the quantum level) are well known, thus in principle one could establish a relation between the movement of the Brownian particle and the underlying microscopic dynamics. However, the complexity involved in such an analysis, makes it impracticable and it is necessary to adopt a stochastic mesoscopic description in order to get useful physical laws. In other cases, like e.g. the stock prices fluctuations, the impossibility of producing a theory based on the physical-chemical state of the traders' brains is evident, so that again the only practicable alternative is a probabilistic description.

The important remarkable point is that notwithstanding (actually "exactly because of") the presence of this large number of stochastic agents, the overall statistics of the process (mean value, variance,...) follows simple reproducible regular laws, given that the analysis is made on appropriate time-scales. The theory of stochastic processes amounts to formulate these time-evolution laws, by substituting the lack of microscopic information with adequate probability hypothesis.

In summary, all processes whose time evolution is usefully analyzed in probabilistic terms are called *stochastic processes*. This notion is quite general and includes scalar and vector processes, with discrete or continuous values. In the following we will concentrate in scalar continuous stochastic processes, but the generalization to the other cases are straightforward.

1.1 Basic definitions

Definition (Stochastic process I $X(t)$)**.** A stochastic process is a collection of random variables defined in the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and parametrized by the time $t \in \mathbb{R}^+$:

$$X(\omega, t) : \ \Omega \times \mathbb{R}^+ \to \mathbb{R}. \tag{1.1}$$

Remark. A stochastic process is a two-variables function. There are hence two possible interpretations:

1. For a given $t^* \in \mathbb{R}^+$, $X(\omega, t^*)$ is a random variable $X(\omega, t^*) : \ \Omega \to \mathbb{R}$;
2. For a given outcome $\omega^* \in \Omega$, the expression $X(\omega^*, t) : \ \mathbb{R}^+ \to \mathbb{R}$ means that as the time passes by, the probability associated to the same outcome ω^* is changing, because the random variable itself is varying.

Remark. Following customary use in probability theory, we will drop in our notations explicit reference to the dependence on ω and indicate the stochastic process as

$$X(t). \tag{1.2}$$

Remark. In term of probability density functions it makes then sense to consider the following distribution density:

$$\begin{aligned} p_{X(t)} &: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R} \\ p_{X(t)}(x, t)\ dx &= \mathbb{P}(x \leq X(t) \leq x + dx). \end{aligned} \tag{1.3}$$

A more precise definition of a stochastic process is given by the following definition.

Definition (Stochastic process II $X(t_1) \ldots, X(t_n)$)**.** Given a partition set $T = \{t_1, t_2, \cdots, t_n\}$, a stochastic process indexed by T is a collection n of random variables $\{X(t_1), \cdots, X(t_n)\}$ (equivalently, a n-dimensional random variable $\mathbf{X}$. If $T \in \mathbb{Z}^+ = \{0, 1, \cdots, \infty\}$ the process is said *discrete*. If $T \in \mathbb{R}^+ = [0, \infty)$ the process is said *continuous*.

The central quantity defining a stochastic process is then the joint probability density function.

Definition (Joint probability density function $p_{X(t_1),\ldots,X(t_n)}$)**.** Given a partition set $T = \{t_1, t_2, \cdots, t_n\}$ and the correspondent stochastic process $\{X(t_1), \cdots, X(t_n)\}$, the joint probability density function of the stochastic process is defined as

$$p_{X(t_1),\ldots,X(t_n)}(x_1, t_1; x_2, t_2; \ldots; x_n, t_n)\ dx_1 \cdots dx_n = \mathbb{P}(x_1 \leq X(t_1) \leq x_1 + dx_1 \cap \cdots \cap x(t_n) \leq X_n \leq x_n + dx_n). \tag{1.4}$$

The time-dependence of the stochastic process in the interval $[t_0, t_0 + \tau]$ is defined by the infinite hierarchy of joint probability density functions obtained $\forall n = 1, 2, \ldots$ and $\forall (t_1, \ldots, t_n)$ with $t_i \in [t_0, t_0 + \tau]$:

$$\begin{aligned} &p_{X(t_1)}(x_1, t_1), \\ &p_{X(t_1),X(t_2)}(x_1, t_1; x_2, t_2), \\ &p_{X(t_1),X(t_2),X(t_3)}(x_1, t_1; x_2, t_2; x_3, t_3), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ &p_{X(t_1),X(t_2),\cdots X(t_n)}(x_1, t_1; x_2, t_2; \cdots; x_n, t_n) \\ &\forall t_i \in [t_0, t_0 + \tau]. \end{aligned} \tag{1.5}$$

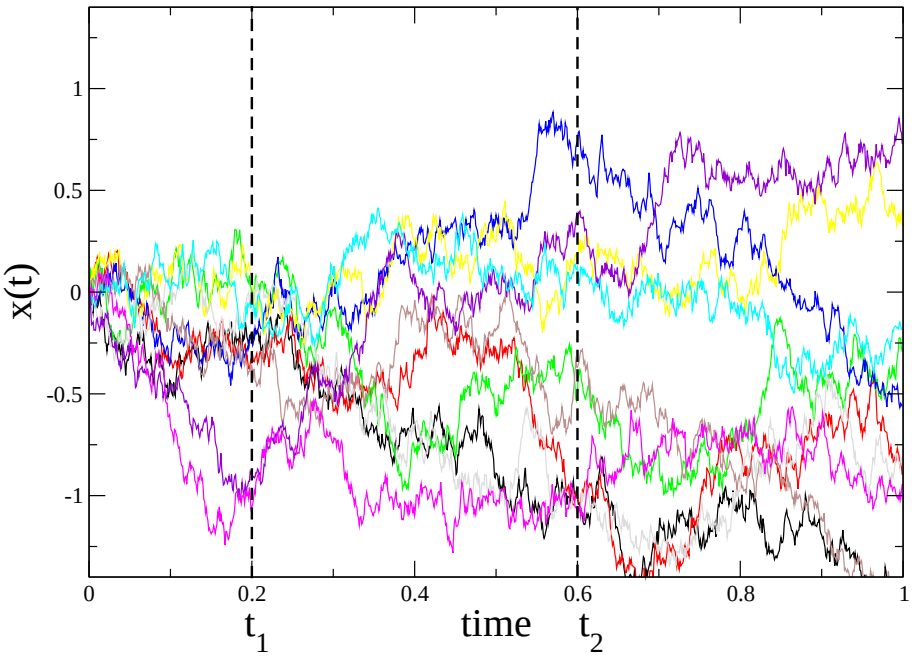

Figure 1.1: Ten different realization of a stochastic process.

For instance, the *two-time correlation function* of the stochastic process is given by

$$\mathbb{E}\{X(t_1)X(t_2)\} \equiv \langle X(t_1)X(t_2)\rangle = \int_{\mathbb{R}^2} dx_1 dx_2 \; x_1 x_2 \; p(x_1,t_1;x_2,t_2). \tag{1.6}$$

Remark (Notation)**.** When it does not generate confusion, we will simplify our notation for the joint probability density function by setting

$$p_{X(t_1),\ldots,X(t_n)}(x_1,t_1;x_2,t_2;\ldots;x_n,t_n) \equiv p(x_1,t_1;x_2,t_2;\ldots;x_n,t_n). \tag{1.7}$$

Remark (Properties of the joint probability density funciton)**.** We remark that the joint probability density function of a stochastic process must satisfy the following properties for any given partition set $T = \{t_1, t_2, \cdots, t_n\}$.

1. $p(x_1,t_1;x_2,t_2;\cdots;x_n,t_n) \geq 0 \quad \forall\{t_1;\cdots;t_n\}$;
2. $\int_{\mathbb{R}^n} dx_1 dx_2 \cdots dx_n \; p(x_1,t_1;x_2,t_2;\cdots;x_n,t_n) = 1 \quad \forall\{t_1;\cdots;t_n\}$;
3. $p(x_1,t_1;x_2,t_2;\cdots;x_n,t_n)$ is a symmetric function with respect to the permutations of the arguments $\{x_1,t_1;\cdots;x_n,t_n\}$;
4. $\int_{\mathbb{R}} dx_n \; p(x_1,t_1;x_2,t_2;\cdots;x_n,t_n) = p(x_1,t_1;x_2,t_2;\cdots;x_{n-1},t_{n-1})$.

The last property is a compatibility relation between the n-point and the $n-1$-point joint probability density functions. It means that the $n-1$-point joint probability density function must be obtained by its n-point analogous through a *reduction*.

If $t_n \to t_{n-1}$ we have

$$\lim_{t_n \to t_{n-1}} p(x_1,t_1;x_2,t_2;\cdots;x_{n-1},t_{n-1};x_n,t_n) = p(x_1,t_1;x_2,t_2;\cdots;x_{n-1},t_{n-1})\; \delta(x_n - x_{n-1}), \tag{1.8}$$

since in this limit the random variables $X(t_n)$ and $X(t_{n-1})$ must coincide.

Notice also that the one-point probability density function $p(x,t)$ is in fact a reduced density function.

Initial conditions
At the initial time $t = t_0$ a stochastic process could begin exactly with the same deterministic initial condition $X(t_0) = x_0$. In such a case, the one-point probability density function is given by $\lim_{t\to t_0} p(x,t) = \delta(x - x_0)$. Conversely, if the initial conditions are themselves a random variable, then $p(x_0, t_0)$ describes their distribution.

Remark (Experimental realization of a stochastic process)**.** In practice, a stochastic process is defined by an empirical determination of its joint probability density function. The fundamental steps can be summarized as follows.

- Prepare many realizations $N \gg 1$ of the same experiment under the same or different initial conditions.
- Evaluate $p(x_1, t_1)\, dx_1$ by calculating the frequency of the outcomes in $[x_1, x_1 + dx_1]$ at time t_1.
- Evaluate $p(x_1, t_1; x_2, t_2)\, dx_1 dx_2$ by calculating the frequency of the outcomes in $[x_1, x_1 + dx_1]$ at time t_1 *and* in $[x_2, x_2 + dx_2]$ at time t_2.
- ...
- Evaluate $p(x_1, t_1; x_2, t_2, \ldots; x_n, t_n)\, dx_1 dx_2 \cdots dx_n$ by calculating the frequency of the outcomes in $[x_1, x_1 + dx_1]$ at time t_1 *and* in $[x_2, x_2 + dx_2]$ at time t_2, *and* ..., *and* in $[x_n, x_n + dx_n]$ at time t_n.

Definition (Conditional probability density function)**.** The conditional probability density function $p(x_n, t_n; \ldots; x_{l+1}, t_{l+1} | x_l, t_l; \ldots; x_1, t_1)$ is defined in the customary way, i.e.,

$$\begin{aligned}
&\mathbb{P}(x_n \le X(t_n) \le x_n + dx_n \cap \cdots \cap x_{l+1} \le X(t_{l+1}) \le x_{l+1} + dx_{l+1} | X(t_l) = x_l, \ldots, X(t_1) = x_1) \\
&\qquad \equiv p(x_n, t_n; \ldots; x_{l+1}, t_{l+1} | x_l, t_l; \ldots; x_1, t_1) dx_n \cdots dx_{l+1} \\
&\qquad = \tfrac{\mathbb{P}(x_n \le X(t_n) \le x_n + dx_n \cap \cdots \cap x_1 \le X(t_1))}{\mathbb{P}(x_l \le X(t_l) \le x_l + dx_l \cap \cdots \cap x_1 \le X(t_1))} \\
&\qquad = \tfrac{p(x_1, t_1; \ldots, x_n, t_n) dx_1 \cdots dx_n}{p(x_1, t_1; \ldots, x_l, t_l) dx_l \cdots dx_1},
\end{aligned} \tag{1.9}$$

where $p(x_1, t_1; \ldots, x_l, t_l) \equiv \int_{\mathbb{R}^{n-l}} dx_{l+1} \cdots dx_n\, p(x_1, t_1; \ldots, x_n, t_n)$.

Hence, the conditional probability density functions inherit by the joint probability density functions analogous properties.

Remark. By iterating the definition of conditional probability (1.9) one gets

$$\begin{aligned}
p(x_1, t_1; \ldots, x_n, t_n) &= p(x_n, t_n | x_{n-1}, t_{n-1}; \ldots; x_1, t_1)\, p(x_{n-1}, t_{n-1} | x_{n-2}, t_{n-2}; \ldots; x_1, t_1) \\
&\times \cdots p(x_2, t_2 | x_1, t_1) p(x_1, t_1).
\end{aligned} \tag{1.10}$$

This means that a stochastic process can be equivalently completely defined in terms of conditional probability density functions.

Example. Let us consider the following stochastic process

$$X(t) = X_1 + X_2 t, \tag{1.11}$$

where X_1 and X_2 are random variables. This process is given by a family of straight lines. Its average and two-point correlation functions are given by

$$\mathbb{E}\{X(t)\} = \mathbb{E}\{X_1\} + \mathbb{E}\{X_2\}\, t \tag{1.12}$$

$$\begin{aligned}
\mathbb{E}\{X(t_1) X(t_2)\} &= \mathbb{E}\{(X_1 + X_2 t_1)(X_1 + X_2 t_2)\} \\
&= \mathbb{E}\left\{X_1^2\right\} + \mathbb{E}\{X_1 X_2\}\,(t_1 + t_2) + \mathbb{E}\left\{X_2^2\right\} t_1 t_2
\end{aligned} \tag{1.13}$$

1.1.1 Conditional expectation

Suppose Y is a random variable measuring the outcome of some random experiment. If one knows nothing about the outcome of the experiment, then the best guess for the value of Y is its expectation value $\mathbb{E}\{Y\}$. If, on the other hadn, one has a complete knowledge of the outcome of the experiment, then one knows the exact value of Y. Conditional expectations deals with making the best guess for Y given some but not all information about the outcome.

Suppose that X and Y are discrete random variables with joint probability density function

$$p(x,y) = Prob\{X = x, Y = y\} \tag{1.14}$$

and marginal probability density functions

$$p_X(x) = \sum_y p(x,y), \qquad p_Y(y) = \sum_x p(x,y). \tag{1.15}$$

The conditional expectation of Y given X, $E\{Y|X\}$ can be then defined as

$$\begin{aligned} E\{Y|X\}(x) &= \sum_y y Prob\{Y = y|X = x\} \\ &= \sum_y y \frac{Prob\{Y = y, X = x\}}{Prob\{X = x\}} \\ &= \frac{\sum_y yp(x,y)}{p_X(x)}. \end{aligned} \tag{1.16}$$

This is of course well defined if $f_X(x) > 0$ i.e. if x is a possible outcome of the experiment.

Example. Suppose that two independent dices are rolled and let X and Y respectively be the value of the first outcome and the sum of the two rolls. Then

$$p(x,y) = \frac{1}{36}, \quad x = 1, 2, \cdots, 6, \quad y = x+1, x+2, \cdots, x+6, \tag{1.17}$$

and

$$E\{Y|X\}(x) = x + \frac{7}{2}. \tag{1.18}$$

In the more general case of $n+1$ discrete random variables with joint probability density

$$p(x_1, x_2, \cdots, x_n, y) = Prob\{X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n, Y = y\}, \tag{1.19}$$

and marginal density with respect to $X_1, \cdots, X_n$ given by

$$p(x_1, \cdots, x_n) = \sum_y p(x_1, \cdots, X_n, y), \tag{1.20}$$

the conditional expectation of Y given $X_1, \cdots, X_n$ is:

$$E\{Y|X_1, \cdots, X_n\}(x_1, \cdots, x_n) = \frac{\sum_y yp(x_1, \cdots, x_n, y)}{p(x_1, \cdots, x_n)} = \sum_y yp(y|x_1, \cdots, x_n). \tag{1.21}$$

Again this is well defined if $x_1, \cdots, x_n$ is a possible outcome for the experiment, i.e., if $p(x_1, \cdots, x_n) > 0$. If X and Y are continous random variables with join density $p(x,y)$ and marginal densities

$$p_X(x) = \int_{\mathbb{R}} p(x,y)dy, \qquad p_Y(y) = \int_{\mathbb{R}} p(x,y)dx, \tag{1.22}$$

then the conditional expectation of Y given $X$is defined in an analogous way

$$E\{Y|X\}(x) = \frac{\int_{\mathbb{R}} yp(x,y)dy}{p_X(x)}. \tag{1.23}$$

which is well defined for $p_X(x) > 0$. Similarly if $X_1 \cdots X_n, Y$ have joint density $p(x_1, \cdots, x_n, y)$,

$$E\{Y|X_1, \cdots, X_n\}(x_1, \cdots, x_n) = \frac{\int_{\mathbb{R}} yp(x_1, \cdots, x_n, y)dy}{p_{X_1,\cdots,X_n}(x_1, \cdots, x_n)} = \int_{\mathbb{R}} yp(y|x_1, \cdots, x_n)dy. \tag{1.24}$$

The conditional expectation $E\{Y|X_1, \cdots, X_n\}$ is characterized by two properties:

1. The value of the random variable $E\{Y|X_1, \cdots, X_n\}$ depends only on the values of $X_1, \cdots, X_n$, i.e., we can write $E\{Y|X_1, \cdots, X_n\} = \phi(X_1, \cdots, X_n)$. Remember that if a random variable ζ can be written as a function of $X_1, \cdots, X_n$ it is called *measurable* with respect to $X_1, \cdots, X_n$

2. Suppose A is an event that depends on $X_1, \cdots, X_n$. For example it could be the event

$$A = \{a_1 \leq X_1 \leq b_1, \cdots, a_n \leq X_n \leq b_n\}. \tag{1.25}$$

   Let I_A denote the indicator function of A, i.e., the random variable which equals 1 if A occurs and 0 otherwise. Then

$$\mathbb{E}\{YI_A\} = \mathbb{E}\left[E\{Y|X_1, \cdots, X_n\}I_A\right]. \tag{1.26}$$

Let us proof the last equality in the case of continuous random variables with density $p(x_1, x_2, \cdots, x_n, y)$ and A the above event.

$$\begin{aligned}
& \mathbb{E}\left[E\{Y|X_1, \cdots, X_n\}I_A\right] \\
= & \int_{a_1}^{b_1} \cdots, \int_{a_n}^{b_n} \int_{\mathbb{R}} E\{Y|X_1 = x_1, \cdots, X_n = x_n\}p(x_1, x_2, \cdots, x_n, y)dydx_n \cdots dx_1 \\
= & \int_{a_1}^{b_1} \cdots, \int_{a_n}^{b_n} \int_{\mathbb{R}} \left[\frac{\int_{\mathbb{R}} zp(x_1, \cdots, x_n, z)dz}{\int_{\mathbb{R}} p(x_1, \cdots, x_n, z)dz}\right] p(x_1, x_2, \cdots, x_n, y)dydx_n \cdots dx_1 \\
= & \int_{a_1}^{b_1} \cdots, \int_{a_n}^{b_n} \int_{\mathbb{R}} zp(x_1, \cdots, x_n, z)dzdx_n \cdots dx_1 \\
= & \mathbb{E}\{YI_A\}.
\end{aligned} \tag{1.27}$$

1.1.2 Stationary processes

A stationary process is process in which $X(t)$ and $X(t + \epsilon)$ have the same statistics for all $\epsilon \in \mathbb{R}$.

Definition (Stationary process)**.** A stochastic process is said stationary if

$$p(x_1, t_1; \ldots; x_n, t_n) = p(x_1, t_1 + \epsilon; \ldots; x_n, t_n + \epsilon) \qquad \forall \epsilon \in \mathbb{R}. \tag{1.28}$$

Specifically, we have that:

- The one-point probability density function does not depend on t:

$$p(x_1, t_1) = p(x_1, t_1 + \epsilon) \qquad \forall \epsilon \in \mathbb{R} \Rightarrow \quad p(x_1, t_1) = f(x_1). \tag{1.29}$$

  As a consequence, the mean value of the stochastic process is constant

$$\mathbb{E}\left\{X(t)\right\} \equiv \langle X(t) \rangle = \eta = \text{cost}. \tag{1.30}$$

- The two-point probability density function depends only onthe difference of the two times:

$$p(x_1, t_1; x_2, t_2) = p(x_1, t_1 + \epsilon; x_2, t_2 + \epsilon) \qquad \forall \epsilon \in \mathbb{R} \Rightarrow \quad p(x_1, t_1; x_2, t_2) = f(x_1, x_2, t_2 - t_1). \tag{1.31}$$

  As a consequence, for the two-time correlation function we have

$$\mathbb{E}\left\{X(t_1)X(t_2)\right\} = \mathbb{E}\left\{X(t_1)X(t_1 + \tau)\right\} \equiv C(\tau), \tag{1.32}$$

  with $\tau = t_2 - t_1$.

There is also a weaker definition of stationarity.

Definition (Stationary process in the weak sense)**.** A stochastic process is said stationary in the weak sense when

$$\begin{aligned}&\mathbb{E}\{X(t)\} = \eta = \text{cost}\\ &\mathbb{E}\{X(t_1)X(t_2)\} = \mathbb{E}\{X(t_1)X(t_1+\tau)\} = C(\tau)\end{aligned} \tag{1.33}$$

Of course, while a stationary process is stationary in the weak sense, a process could be stationary in the weak sense but not stationary.

Remark (Deducing the statistics from a stationary time series)**.** Sometimes, it is not possible to average over several realizations of a stochastic process but just one long random path (i.e. one $X(t)$) is available. This is for example the case of time series for a wether variable or for the light intensity of a star. If however, the process is stationary it is possible to subdivide the time series in N sub-series each of equal length T and then average over them.

1.1.3 Ergodic processes

Let us consider a stationary stochastic process $\{X(t)\}_{t\geq 0}$ and let $x(t)$ a given realisation of the process $X(t)$. If the arbitrarely chosen $x(t)$ contains all the statistics informations of the process $X(t)$, the process itself is *ergodic*. To more precise let us define first the time average of a stochastic process.

Definition. Time average of $X(t)$ For a given realisation $x(t)$ ed time interval $[t-T/2, t+T/2]$, we define the time average over the interval as

$$\boxed{\overline{X}_T(t;x) \equiv \frac{1}{T}\int_{t-T/2}^{t+T/2} x(t')dt'} \tag{1.34}$$

Clearly the above definition depends on T, t and also on the given realisation $x(t)$. However, in the limit $T \to \infty$, one can define the time average of the process that does not depend either on t or on T

$$\overline{X}(x) = \lim_{T\to\infty} \overline{X}_T(t;x) \tag{1.35}$$

In principle the above definition depends still on the given realisation x. If it doesn't we finally have

$$\overline{X}(x) = \overline{X} = \mathbb{E}\{X\} \tag{1.36}$$

and the process is said to be *ergodic in mean*. More generally the process $X(t)$ is *ergodic* is the equivalence between the time and ensemble averages holds not only for the first moment but for all the moments of the distribution $p(x_1, t_1, \cdots, x_n, t_n)$.

1.2 Harmonic analysis of a stationary process

A given realization $x(t)$ of a generic stochastic process $\{X(t)\}_{t\geq 0}$ (stationary or not) is not, a priori, neither a periodic function that can be expanded in Fourier series nor a L^2 function for which a Fourier transform exists. This is in particular true for stationary processes where a generic realization $x(t)$ does not go to zero as $t \to \infty$. We can neverthenless define a Fourier transform of $x(t)$ by using the following trick. Suppose we have observed a stochastic process within a time interval $[0, T]$, arbitrary large but finite. The Fourier transform of a given realization $x(t)$ within this interval can be defined for the function

$$X_T(t) = \begin{cases} x(t) & t \in [0,T] \\ 0 & \text{otherwise} \end{cases} \tag{1.37}$$

as

$$\tilde{x}(\omega) = \int_{\mathbb{R}} x_T(t) e^{i\omega t} dt = \int_0^T x(t) e^{i\omega t} dt. \tag{1.38}$$

The inverse is then given by

$$x_T(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{x}(\omega) e^{-i\omega t} d\omega \tag{1.39}$$

At the end of the calculation the limit $T \to \infty$ can be taken. Since the chosen realisation is arbitrary we can formally define the Fourier transform of the stochastic process $\{X(t)\}_{t\geq 0}$ as

$$\tilde{X}(\omega) = \int_{\mathbb{R}} X(t) e^{i\omega t} dt \tag{1.40}$$

and

$$X(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{X}(\omega) e^{-i\omega t} d\omega. \tag{1.41}$$

1.2.1 Fourier Series of a stochastic process

Similarly, since a generic realization of a stochastic process is not a periodic function, we can make it periodic on a large but finite interval $[0, T]$ as follows. For T fixed we consider the realization $y(t)$ obtained by repeating the original realisation $x(t)$ on the interval $[0, T]$ on all the intervals $[T, 2T], [2T, 3T] \cdots$ We can then define the Fourier series of $x(t)$ as

(1.42)

$$x(t) = \sum_{n=-\infty}^{+\infty} a_n e^{-i\omega_n t} \qquad 0 \geq t \geq T \tag{1.43}$$

where $\omega_n = 2\pi n/T$, $n = 0, \pm 1, \pm 2, \ldots$, and a_n is a complex number given by

$$a_n = \frac{1}{T} \int_0^T x(t) e^{i\omega_n t} dt. \tag{1.44}$$

Since for every realization $x(t)$ we define in such a way a countably infinite set of complex numbers $\{a_n\}_{n\in\mathbb{Z}}$, the stochastic process $X(t)$ within the interval $[0, T]$ is in fact expressed by a countably infinite set of random variables $\{A_n\}_{n\in\mathbb{Z}}$. We can then define formally the Fourier transform of the stochastic process $X(t)$ as

$$X(t) = \sum_{n=-\infty}^{+\infty} A_n e^{-i\omega_n t} \qquad 0 \geq t \geq T \tag{1.45}$$

with

$$A_n = \frac{1}{T} \int_0^T dt X(t) e^{i\omega_n t}. \tag{1.46}$$

It is important to notice that, being T fixed, the following relation

$$\boxed{A_n = \frac{1}{T} \tilde{X}(\omega).} \tag{1.47}$$

From the previous considerations we can define the expectation value of A_n as

$$\mathbb{E}\{A_n\} = \frac{1}{T} \int_0^T \mathbb{E}\{X(t)\} e^{i\omega t} dt. \tag{1.48}$$

If the process is stationary $\mathbb{E}\{X(t)\} = \mathbb{E}\{X\} \equiv \eta$ and

$$\mathbb{E}\{A_n\} = \frac{\mathbb{E}\{X\}}{T}\int_0^T e^{-iw_n t}\,dt = \begin{cases} 0 & \text{if } n \neq 0 \\ \mathbb{E}\{X\} = \eta & \text{if } n = 0, \end{cases} \tag{1.49}$$

Since all the realisation a_0 of A_0 are equal to $\mathbb{E}\{X\}$ A_0 is practially a deterministic variable and we can consider instead the centred stochastic process $X(t) - \mathbb{E}\{X\}$. By construction this process can be expanded in Fourier series (in the sense explained above) with random coefficients A_n having $\mathbb{E}\{A_n\} = 0$, for all $n = 0, \pm 1 \cdots$. For this process we can define the two-points correlation function as

$$C_X(t,\tau) \equiv \mathbb{E}\{X(t+\tau)X^*(\tau)\} \tag{1.50}$$

and since

$$X(t) = \sum_{n=-\infty}^{\infty} A_n e^{-i\omega_n t} \tag{1.51}$$

we have

$$C_X(t,\tau) = \sum_{n=-\infty}^{n=\infty}\sum_{n'=-\infty}^{\infty} \mathbb{E}\{A_n A_{n'}^*\} e^{-i(\omega_n - \omega_{n'})t} e^{-i\omega_n \tau}. \tag{1.52}$$

On the other hand, if the process is stationary (even in the weak sence), the correlation function cannot depend on t and the following relation must hold

$$\boxed{\mathbb{E}\{A_n A_{n'}^*\} = \mathbb{E}\{|A_n|^2\}\delta_{nn'}.} \tag{1.53}$$

In other words, for a stationary process, there are no correlations between different modes of the Fourier expansion.

1.2.2 Wiener-Khintchine theorem

Let $\{X(t)\}_{t\geq 0}$ be a stationary process and consider a filter that select, among all the possible frequencies ω_n , the ones within the interval $[\omega, \omega + \Delta\omega]$. We can define the average intensity $\sigma(\omega)$ in this frequency band as

$$\sigma(\omega)\Delta\omega = \sum_{\omega_n \in [\omega, \omega+\Delta\omega]} \mathbb{E}\{|A_n|^2\}. \tag{1.54}$$

Clearly the number of modes m within the interval $[\omega, \omega + \Delta\omega]$ is simply given by

$$m = \frac{\Delta\omega}{2\pi/T} = \frac{T\Delta\omega}{2\pi}. \tag{1.55}$$

We now take the limit $T \to \infty$ and $\Delta\omega \to 0$. For very large observation times T the frequencies spectrum tends to be continuous $A_n \mapsto A(\omega)$. This, in addition, enables us to take the filtering interval $\Delta\omega$ very small. Under these conditions, the sum in Eq. (1.54) can be approximated by $m\,\mathbb{E}\{|A(\omega)|^2\}$ and we get

$$\sigma(\omega) \equiv \lim_{T\to\infty} \frac{1}{\Delta\omega}\, m\, \mathbb{E}\{|A(\omega)|^2\} \tag{1.56}$$

$$= \lim_{T\to\infty} \frac{T}{2\pi}\, \mathbb{E}\{|A(\omega)|^2\} \tag{1.57}$$

Sometimes the quantity $\sigma(\omega)$ is replaced by $S(\omega) = 2\pi\sigma(\omega)$. If, in addition, we consider the relation (1.47) we finally have

$$\boxed{S(\omega) = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\{|\tilde{X}(\omega)|^2\}} \tag{1.58}$$

To make an example: If $X(t)$ is the noise voltage between two terminals in an electrical network, then $S(\omega)$ is the intensity of the noise "heard" by filtering the frequencies to a narrow bandwidth $\Delta\omega$ around ω.

Note that relation (1.58) allows to establish the continuum limit of eq. (1.53), namely

$$\mathbb{E}\{\tilde{X}(\omega)\tilde{X}^*(\omega')\} = 2\pi\delta(\omega-\omega')S(\omega). \tag{1.59}$$

We can now establish the theorem by simply applying the above procedure to the expression

$$C_X(\tau) = \sum_{n=-\infty}^{\infty} \mathbb{E}\{|A(\omega_n)|^2\}e^{-i\omega_n\tau} \tag{1.60}$$

Indeed in the limits $T\to\infty$, $\Delta\omega\to 0$, and using $A_n = \frac{1}{T}\tilde{X}(\omega_n)$ we obtain

$$C_X(\tau) = \lim_{T\to\infty}\frac{1}{2\pi T}\int_{\mathbb{R}} \mathbb{E}\{|\tilde{X}(\omega)|^2\}e^{-i\omega\tau}d\omega \tag{1.61}$$

and from (1.58) we finally get

$$\boxed{C_X(\tau) = \frac{1}{2\pi}\int_{\mathbb{R}} S(\omega)e^{-i\omega\tau}d\tau} \tag{1.62}$$

Similarly

$$\boxed{S(\omega) = \int_{\mathbb{R}} C_X(\tau)e^{i\omega\tau}d\tau} \tag{1.63}$$

Eqs. (1.62) and (1.63) correspond to the *Wiener-Khintchine theorem* (Wiener 1930 and Khintchine 1934), a result that relates the power spectrum to the autocorrelation function of the stochastic process, showing that one is the Fourier transform of the other. Each of these two quantities contain the same information about the process.

Note that, if the stochastic process is real, the power spectrum is real and even, $S(-\omega) = S^*(\omega) = S(\omega)$. In such a case, a simplified definition of the power spectrum is

$$S(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} d\tau\ \cos(\omega\tau)\ C(\tau) = \frac{1}{\pi}\int_0^{+\infty} d\tau\ \cos(\omega\tau)\ C(\tau). \tag{1.64}$$

1.3 Classification of stochastic processes

1.3.1 Completely random stochastic process

Definition (Completely random stochastic process)**.** A completely random stochastic process is characterized by the property

$$p(x_n,t_n|x_{n-1},t_{n-1},\ldots;x_1,t_1) = p(x_n,t_n). \tag{1.65}$$

In other words, the probability of $X(t_n)$ conditioned to the previous $n-1$ events ($n\geq 2$) does not depend on the outcomes $x_i = X(t_i)$, with $t_i < t_n$. Hence, the process is completely memoryless. We have then that the joint probability density function is given by

$$p(x_1,t_1;\ldots;x_n,t_n) = p(x_1,t_1)\cdots p(x_n,t_n), \tag{1.66}$$

and the whole process is determined by $p(x_1,t_1)$.

1.3.2 Markov Processes

A very important class of stochastic processes is the one of Markov processes whose definition goes as follow:

Definition (Markov process)**.** A stochastic process $X(t) \in \mathbb{R}$ is called Markov process if $\forall t_1 < t_2 < \cdots < t_n$ and $\forall n$, the conditional probability $p(x_n, t_n|x_1, t_1; x_2, t_2; \cdots; x_{n-1}, t_{n-1})$ satisfies the property:

$$p(x_n, t_n|x_1, t_1; x_2, t_2; \cdots; x_{n-1}, t_{n-1}) = p(x_n, t_n|x_{n-1}, t_{n-1}). \tag{1.67}$$

The above definition is equivalent to say that the event characterized by $\{x_n \leq X(t_n) \leq x_n + dx_n\}$ depends only on the previous event $\{X(t_{n-1}) = x_{n-1}\}$. In other words a Markov process does not depend on its whole history (stochastic process with no memory).

Lemma 1.3.1. A Markovian stochastic process is completely determined by the one-point density probability $p(x, t)$ and by the conditional probability $p(x_2, t_2|x_1, t_1)$.

Proof. We know that for a generic stochastic process the n-point joint distribution is given by

$$\begin{aligned} &p(x_1, t_1; x_2, t_2; \cdots; x_n, t_n) = p(x_1, t_1)p(x_2, t_2|x_1, t_1)p(x_3, t_3|x_2, t_2; x_1, t_1) \\ &\cdots p(x_{n-1}, t_{n-1}|x_{n-2}, t_{n-2}; \cdots; x_1, t_1)p(x_n, t_n|x_{n-1}, t_{n-1}; \cdots; x_1, t_1). \end{aligned} \tag{1.68}$$

By applying the Markov property to the previous equation we then obtain the lemma:

$$\begin{aligned} &p(x_1, t_1; x_2, t_2; \cdots; x_n, t_n) = p(x_1, t_1)p(x_2, t_2|x_1, t_1)p(x_3, t_3|x_2, t_2) \\ &\cdots p(x_{n-1}, t_{n-1}|x_{n-2}, t_{n-2})p(x_n, t_n|x_{n-1}, t_{n-1}). \end{aligned} \tag{1.69}$$

□

Note. For Markov processes the $1|1$ conditional probability $p(x_{k+1}, t_{k+1}|x_k, t_k)$ is also known as *transition probability*.

Lemma 1.3.2. Let $X(t)$ be a stochastic process. Show that if for $t_1 < t_2$ the increment $X(t_2) - X(t_1)$ does not depend on $X(t)$, $\forall t \leq t_1$, then the process $X(t)$ is a Markov process.

Proof. **Exercise.** □

A consequence of the above result is the following theorem:

Theorem 1.3.3. If $X(t)$ is a stochastic process with independent increments $X(t_{k+1}) - X(t_k)$ and $X(t = 0) = 0$, then $X(t)$ is a Markov process.

Proof. Since the increments $X(t_{k+1}) - X(t_k)$ are independent, in particular the increment $X(t_2) - X(t_1)$ does not depend on

$$X(t) - X(0) = X(t), \qquad \forall t \leq t_1. \tag{1.70}$$

We can then use lemma 1.3.2 to obtain the result. □

Remark. From the definition of Markov process and conditional probability it turns out that

$$E\{X(t_n)|X(t_{n-1}), \cdots, X(t_1)\} = E\{X(t_n)|X(t_{n-1})\} \tag{1.71}$$

Indeed

$$\begin{aligned} E\{X(t_n)|X(t_{n-1}), \cdots, X(t_1)\} &= \int_{\mathbb{R}} x(t_n)p(x_n, t_n|x_{n-1}, t_{n-1}, \cdots, x_1, t_1)dx_n \\ &= \int_{\mathbb{R}} x(t_n)p(x_n, t_n|x_{n-1}, t_{n-1})dx_n = \mathbb{E}\{X(t_n)|X(t_{n-1})\} \end{aligned} \tag{1.72}$$

1.3.3 Chapman-Kolmogorov equation

Theorem 1.3.4 (Chapman-Kolmogorov equation)**.** Let $X(t)$ a Markov process. Then the $1|1$ conditional density probability $p(x_3, t_3|x_1, t_1)$ satisfies the relation:

$$\boxed{p(x_3, t_3|x_1, t_1) = \int_{\mathbb{R}} p(x_3, t_3|x_2, t_2)p(x_2, t_2|x_1, t_1)dx_2} \tag{1.73}$$

Proof. From the integration rule on right variables one gets

$$p(x_3, t_3|x_1, t_1) = \int_{\mathbb{R}} p(x_3, t_3|x_2, t_2; x_1, t_1)p(x_2, t_2|x_1, t_1)dx_2. \tag{1.74}$$

On the other hand, since $X(t)$ is a Markov process,

$$p(x_3, t_3|x_2, t_2; x_1, t_1) = p(x_3, t_3|x_2, t_2) \tag{1.75}$$

and eq. (1.73) is recovered. □

The integral equation (1.73) is fundamental for the theory of Markov processes. Its importance comes from the fact that it is a closed equation for the two-point conditional probability density function. Note that the expression (**??**) for the one-point probability density function of a Markov process is instead not closed.

Remark. A possible intuitive interpretation of the Chapman-Kolmogorov equation is the following: The process started in $\{x_1, t_1\}$ reaches the event $\{x_3, t_3\}$ by passing through any of the possible events described by $X(t_2)$. Indeed the integration over x_2 does represent the sum over all possible ways, at time t_2 to reach x_3 at time t_3.

Note. Usually in physics one can say that an evolution has a markovian character if it can be described by a $1|1$ conditional probability satisfying the Chapman-Kolmogorov equation. However, from a more rigorous point of view, in order to say that an evolution has a markovian character it is necessary to check that such evolution satisfy also eq. (1.67) for any value of n. This is, of course, not possible experimentally.

Definition (Homogeneous Markov processes)**.** A stochastic Markov process is *homogenous* (in time) if

1. $p(x_2, t_2|x_1, t_1) = p(x_2, t_2 - t_1|x_1)$.

Remembering the definition of a generic stationary stochastic process we can define it for Markov processes as follows:

Definition. A stochastic Markov process is *stationary* if

1. $p(x_2, t_2|x_1, t_1) = p(x_2, t_2 - t_1|x_1)$.
2. $p(x, t) = p(x)$.

Clearly a stationary Markov process is also a homogeneous one while the contrary is not true.

Example (Deterministic equation)**.** Let us consider a first order differential equation

$$\frac{dx}{dt} = F(x(t), t), \tag{1.76}$$

and let $\phi(x_0, t - t_0)$ the flux of the equation that gives the trajectory $x(t) = \phi(x_0, t - t_0)$ corresponding to the initial condition $x(t_0) = x_0$. Let us choose some points of that trajectory, for

example $\{x_0, t_0; x_1, t_1; \cdots; x_n, t_n\}$. Since the initial condition determines completely the solution, we have:

$$\begin{aligned} x_1 &= \phi(x_0, t_1 - t_0), \\ x_2 &= \phi(x_0, t_2 - t_0) = \phi(x_1, t_2 - t_1), \\ &\vdots \\ x_n &= \phi(x_0, t_n - t_0) = \phi(x_{n-1}, t_n - t_{n-1}). \end{aligned} \tag{1.77}$$

Since the particle starting from $\{x_0, t_0\}$ has to pass through all the points $\{x_i, t_i\}_{i=1}^n$, we have by definition:

$$\begin{aligned} p(x_1, t_1; \cdots; x_n, t_n) &= \delta(x_1 - \phi(x_0, t_1 - t_0))\delta(x_2 - \phi(x_0, t_2 - t_0)) \times \\ &\times \delta(x_3 - \phi(x_0, t_3 - t_0)) \cdots \delta(x_n - \phi(x_0, t_n - t_0)), \end{aligned} \tag{1.78}$$

and from eq. (1.77) one gets

$$\begin{aligned} p(x_1, t_1; \cdots; x_n, t_n) &= \delta(x_1 - \phi(x_0, t_1 - t_0))\delta(x_2 - \phi(x_1, t_2 - t_1)) \times \\ &\times \delta(x_3 - \phi(x_2, t_3 - t_2)) \cdots \delta(x_n - \phi(x_{n-1}, t_n - t_{n-1})). \end{aligned} \tag{1.79}$$

By inserting the definitions

$$p(x_k, t_k | x_{k-1}, t_{k-1}) = \delta(x_k - \phi(x_{k-1}, t_k - t_{k-1})), \tag{1.80}$$

$$p(x_1, t_1) = \delta(x_1 - \phi(x_0, t_1 - t_0)), \tag{1.81}$$

in eq. 1.79 we then obtain

$$p(x_1, t_1; \cdots; x_n, t_n) = p(x_1, t_1)p(x_2, t_2|x_1, t_1)p(x_3, t_3|x_2, t_2) \cdots p(x_n, t_n|x_{n-1}, t_{n-1}), \tag{1.82}$$

i.e. the Markov property.

Note. The classical mechanics is then a vectorial Markov process given that the pairs $\{q(t), p(t)\}$ are the solutions of the Hamilton equations

$$\dot{q}(t) = \frac{\partial}{\partial p} H(q, p) \tag{1.83}$$

$$\dot{p}(t) = -\frac{\partial}{\partial q} H(q, p), \tag{1.84}$$

$$\tag{1.85}$$

with initial conditions $q(0) = q_0$ and $\dot{q}(0) = v_0$. It is interesting to notice that if one considers just the time evolution of the variable q, the mechanics process loose the Markov property since the value of just the position does not determine completely the trajectory. This is actually a general property of Markov process, i.e., a process that is not Markovian for a set of variables can become Markovian by adding another (or more) variables.

Definition (Non-Markovian process)**.** A process which is not completely random nor Markovian is said a non-Markovian stochastic process.

A non-Markovian process could be characterized by long-time memory effect. A strategy to handle non-Markovian processes is to introduce auxiliary variables in order to obtain a Markov process of higher dimension and then to project back the results for original process.

1.3.4 Markov sequences

Up to now we have considered Markov processes in which both time and $X(t)$ were continues variables i.e. with values in $\mathbb{R}$. It is possible, however, to consider a sequence of random variables

$$X_1, X_2, \cdots, X_n \tag{1.86}$$

in which $t \in \mathbb{N}$. Similarly to the Markov process theMarkov sequence can be defined as follows:

Definition (Markov sequence)**.** A sequence of random variables is markovian if, for any n we have:

$$P(x_n|x_{n-1}, x_{n-2}, \cdots, x_1) = P(x_n|x_{n-1}), \tag{1.87}$$

namely, if the distribution of X_n conditioned by the sequence

$$X_{n-1}, X_{n-2}, \cdots, X_1 \tag{1.88}$$

is equal to the probability distribution of X_n conditioned only by x_{n-1}. If the random variables considered are continues the above definition can be extended to the density distribution p:

$$p(x_n|x_{n-1}, x_{n-2}, \cdots, x_1) = p(x_n|x_{n-1}). \tag{1.89}$$

Again we will have:

$$p(x_1, x_2, \cdots, x_n) = p(x_n|x_{n-1})p(x_{n-1}|x_{n-2}) \cdots p(x_2|x_1)p(x_1). \tag{1.90}$$

Definition (Homogeneous Markov sequence)**.** A Markov sequence is called *homogeneous* if the conditional density

$$p(x_n|x_{n-1}) \tag{1.91}$$

does not depend on n.

Similarly to Markov processes, the transition probability of a Markov sequence satisfies the Champan-Kolmogorov equation:

$$p(x_n|x_s) = \int_{\mathbb{R}} p(x_n|x_r)p(x_r|x_s)dx_r, \qquad \text{for any integers} \quad n > r > s. \tag{1.92}$$

1.3.5 Markov chains

If, in addition to the discrete nature of the time, also the random variables X_k can assume only discrete values such as, for example,

$$a_1, a_2, \cdots, a_n. \tag{1.93}$$

and

$$P\{X_n = a_{i_n}|X_{n-1} = a_{i_{n-1}}, \cdots, X_1 = a_{i_1}\} = P\{X_n = a_{i_n}|X_{n-1} = a_{i_{n-1}}\}. \tag{1.94}$$

the sequence X_n is called *Markov chain.*

Example. Given the sequence $X_1, X_2, \cdots, X_n$ of independent random variables with density $p_{X_n}(x_n) = p(x)$ we can form the sequence:

$$Y_1 = X_1, Y_2 = X_1 + X_2, \cdots, Y_n = X_1 + \cdots + X_n. \tag{1.95}$$

This sequence is a Markov sequence. Indeed, since from the definition of Y_k we have

$$p(y_1, y_2, \cdots, y_n) = p_1(y_1)p_2(y_2 - y_1) \cdots p_n(y_n - y_{n-1}) \tag{1.96}$$

giving

$$p(y_n|y_{n-1}, \cdots, y_1) = \frac{p(y_1, \cdots, y_n)}{p(y_1, \cdots, y_{n-1})} = p_n(y_n - y_{n-1}). \tag{1.97}$$

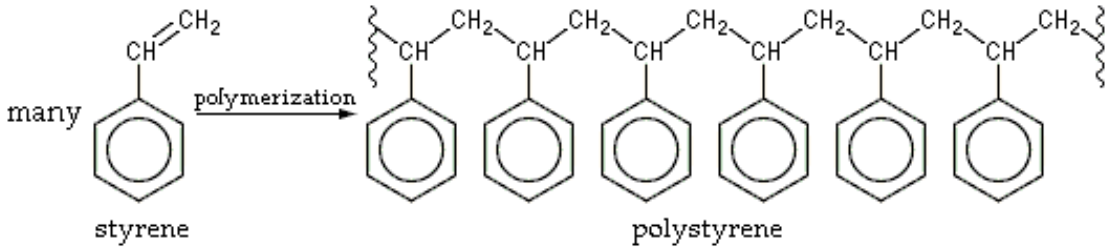

Figure 1.2: Formation of Polystyrene

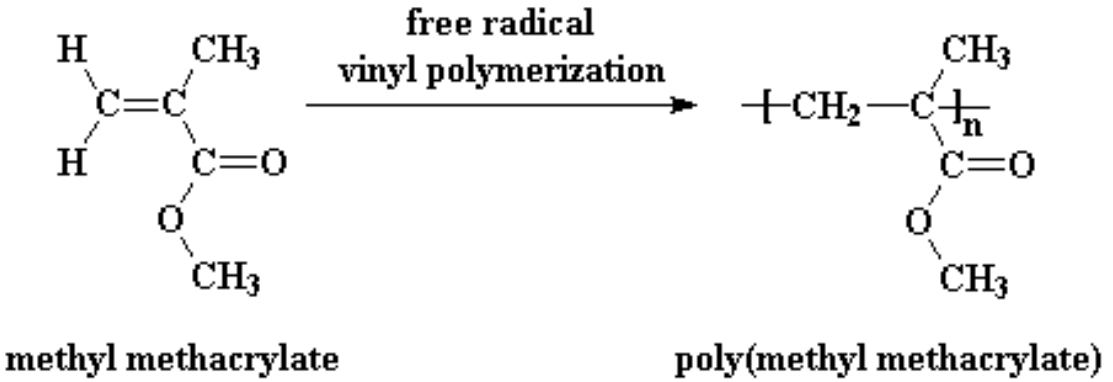

Figure 1.3: Formation of PMMA

On the other hand the last expression is independent on $y_{n-2}, \cdots, y_1$ and the Y_n is a Markov sequence. Note that if

$$\mathbb{E}\{X_n\} = 0, \quad \text{then} \quad \mathbb{E}\{Y_n\} = 0. \tag{1.98}$$

Moreover, since

$$Y_n = X_n + Y_{n-1}, \tag{1.99}$$

we have

$$\mathbb{E}\{Y_n|Y_{n-1}\} = \mathbb{E}\{X_n|Y_{n-1}\} + \mathbb{E}\{Y_{n-1}|Y_{n-1}\} = Y_{n-1}, \tag{1.100}$$

since X_n is independent on Y_{n-1} and $\mathbb{E}\{X_n\} = 0$. Hence,

$$\mathbb{E}\{Y_n|Y_{n-1}, \cdots, Y_1\} = Y_{n-1}, \quad \forall n. \tag{1.101}$$

A sequence with this property is a *martingale*.

1.4 Ideal polymer models as markov sequences

Polymers are in general made by assembling a large number of elementary units called *monomers*. The simplest way to assemble these monomers is to connect them sequencially forming a chain. This gives the topology of a *linear polymer*. There are also situations in which monomers are connected following different graph topology resulting for example, to *polymer rings*, *star polymers* and *branched polymers*. If all the monomers in the polymer are equal (or better to say they have similar chemical properties) one talks about *homopolymer*. There are situations, however in which different monomer types concur to form the polymer. This is the more general case of *heteropolymers*. Note that important biological polymers such as DNA, RNA and proteins are

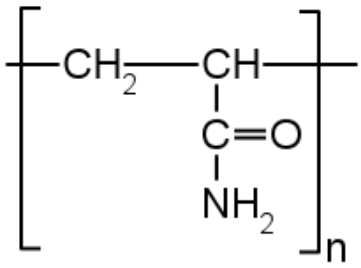

Figure 1.4: Elementary unit for the Polyacrylamide

heteropolymers. Another interesting class of heteropolymer are the *block copolymers* where the chain (linear polymer) is made by two blocks of homopolymers made by by different monomer units. Because monomers can be quite big (think for example to the aminoacids in proteins) contiguous monomers along the chain can interact quite easily by steric hyndrance. This interaction makes the chain sufficiently rigid within a given length scale. In a more statistical sense one can say that in its spatial arrangement the chain tends to follow a given direction for an average number of monomers defining the *persistence length* of the polymer. If the persistence length is sufficiently long the polymer is said to be *semiflexible* or even rigid. Otherwise the polymer is *flexible*. An important example of semiflexible polymer is the DNA that in the double-stranded form has a persistence length of roughly 150 bais pairs. For length scales much larger than the persistence length the polymer is highly flexible and in equilibrium, in free space, it can assume a large number of spatial configurations. In general the number of these configurations grows *exponentially* with N giving rise to an extensive (i.e. proportional to N) *configurational entropy*

$$S_N \propto \log c_N \tag{1.102}$$

This high number of configurations and the fact that in general a polymer is in solution and hence in contact with a heat bath at a given temperature justifies the use of statistical mechanical approaches to these systems.

Since we are interested to study the behaviour of long macromolecules we can consider a *coarse-grained* version of the polymer in which all the chemical details are neglected by considering each monomer as a point in space described by the position vector $\vec{R}_i$

For a polymer made by $N+1$ monomers we have a system with $3N$ degrees of freedom and each *spatial configuration* C is represented by the set $\{\vec{R}_1, \vec{R}_2, \cdots, \vec{R}_N\}$. One is in particular interested in studying the probability of a configuration

$$p_N(\vec{R}_1, \vec{R}_2, \cdots, \vec{R}_N) \tag{1.103}$$

If, among all the possible interactions between monomers and solvent molecules-monomers, we consider only to keep the covalent bonds interactions that is responsible of the connectivity, a configuration $\{\vec{R}_1, \vec{R}_2, \cdots, \vec{R}_N\}$ can be see as a realization of a stochastic process in $\mathbb{R}^3$. A very simple model in this respect is the so called *freely-jointed chain*. It is a model of ideal (i.e. no excluded volume interactions) chain in which the chain is build up by joining segments of equal length b in a completely random fashion. In particular given the chain up to $N-1$ segments the next segment (step) is chosen with length b and with direction uniformly on the surface of the sphere with radius b. Since the system is translational invariant, we can assume the first monomer to be fixed at the origin, i.e. $\vec{R}_0 = \vec{0}$. From the theory of stochastic processes we can write

$$p(\vec{R}_1, \vec{R}_2, \cdots, \vec{R}_N) == p(\vec{R}_N|\vec{R}_{N-1}\cdots\vec{R}_1)p(\vec{R}_{N-1}|\vec{R}_{N-2}\cdots\vec{R}_1)\cdots p(\vec{R}_2|\vec{R}_1). \tag{1.104}$$

Let us consider the generic conditional distribution $p(\vec{R}_{k+1}|\vec{R}_k\cdots\vec{R}_1)$. Clearly the new vector position $\vec{R}_{k+1}$ depends only on the previous position $\vec{R}_k$ and on the way in which the next step is chosen in space. This gives

$$p(\vec{R}_{k+1}|\vec{R}_k\cdots\vec{R}_1) = p(\vec{R}_{k+1}|\vec{R}_k) \tag{1.105}$$

i.e. the ideal polymer is a Markov sequence. In the particular case of the FJC we have

$$p(\vec{R}_{k+1}|\vec{R}_k) = \frac{1}{4\pi b^2}\delta\left(|\vec{R}_{k+1} - \vec{R}_k| - b\right) \tag{1.106}$$

where the factor $1/4\pi b^2$ is for normalization. Notice that the transition probability above does not depend on k, i.e. the Markov sequence is homogeneous. The probability of a given polymer configuration with N monomers is then given by

$$\begin{aligned} p(\vec{R}_1, \vec{R}_2, \cdots, \vec{R}_N) &= p(\vec{R}_N|\vec{R}_{N-1})p(\vec{R}_{N-1}|\vec{R}_{N-2})\cdots p(\vec{R}_2|\vec{R}_1) \\ &= \frac{1}{4\pi b^2}\delta\left(|\vec{R}_N - \vec{R}_{N-1}| - b\right)\cdots\frac{1}{4\pi b^2}\delta\left(|\vec{R}_2 - \vec{R}_1| - b\right). \end{aligned} \tag{1.107}$$

Now let us consider the elementary step vectors $\vec{r}_k \equiv \vec{R}_k - \vec{R}_{k-1}$. Those are somehow the *increments* of the Markov sequence $\{\vec{R}_1, \cdots, \vec{R}_N\}$. It is easy to see that the process of the increments has joint probability distribution

$$p(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N) = \left[\frac{1}{4\pi b^2}\delta\left(|\vec{r}| - b\right)\right]^N, \tag{1.108}$$

i.e. the process of the increments is a completely random process. In other words the increments of the process are independent random variables. An important quantity that characterizes a polymer conformation (for linear polymers) is the *end-to-end vector* $\vec{R}_e = \vec{R}_N - \vec{R}_0$. This can be easily defined as a function of the increments $\{\vec{r}_i\}$

$$\vec{R}_e = \sum_{i=1}^{N} \vec{r}_i = h(\vec{r}_1, \cdots, \vec{r}_N). \tag{1.109}$$

It is interesting to study the distribution function of the end-to-end distance. From the theory of function of random variables we now

$$p(\vec{R}_e; N) = \int_{\mathbb{R}^N} d\vec{r}_1 \cdots d\vec{r}_N \delta(\vec{R}_e - \sum_{i=1}^{N} \vec{r}_i) p(\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_N) \tag{1.110}$$

$$\begin{aligned}
\int_{\mathbb{R}^3} d^3\vec{r} \frac{\delta(|\vec{r}| - b)}{4\pi b^2} e^{-i\vec{k}\cdot\vec{r}} &= \frac{1}{4\pi b^2} \int_0^\infty dr r^2 \delta(r - b) \int_0^\pi d\theta \sin\theta e^{-i|\vec{k}|r\cos\theta} \int_0^{2\pi} d\phi \\
&= \frac{1}{2} \int_{-1}^{1} d(\cos\theta) e^{-i|\vec{k}|b\cos\theta} \\
&= \frac{\sin|\vec{k}|b}{|\vec{k}|b}.
\end{aligned} \tag{1.111}$$

The end-to-end distribution is then given by

$$G(\vec{R}, N) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} d^3\vec{\kappa} \left(\frac{\sin|\vec{k}|b}{|\vec{k}|b}\right)^N e^{i\vec{k}\cdot\vec{R}}. \tag{1.112}$$

Using the spherical coordinates system again such that ϕ is the rotation angle around $\vec{R}$ (fixed in this integration), one can write (using the relation $\vec{k}\cdot\vec{R} = |\vec{k}||\vec{R}|$) we obtain

$$\begin{aligned}
G(\vec{R}, N) &= \frac{1}{(2\pi)^2} \int_0^\pi \int_0^\infty e^{-ikR\cos\theta} \left[\frac{\sin kb}{kb}\right]^N k^2 \sin\theta dk d\theta \\
&= \frac{1}{(2\pi)^2} \int_0^\infty \left[\frac{\sin kb}{kb}\right]^N k^2 \left[\int_{-1}^{1} e^{-ikRs} ds\right] dk
\end{aligned} \tag{1.113}$$

where we have used $k \equiv |\vec{k}|$, $R = |\vec{R}|$ and $s = \cos\theta$. Hence

$$\boxed{G(\vec{R}, N) = \frac{1}{2\pi^2 R} \int_0^\infty k \sin(kR) \left[\frac{\sin kb}{kb}\right]^N dk} \tag{1.114}$$

The integral above is not easy to perform but we can look at the asymptotic formula in the case of $N \to \infty$

To simplify notations let us write formula 1.114 as follows

$$G(\vec{R}, N) = \frac{1}{2\pi^2 R} \int_0^\infty k \sin(kR) e^{N\psi(k)} dk \tag{1.115}$$

where

$$\psi(k) = \log\left[\frac{\sin(kb)}{kb}\right] \tag{1.116}$$

In the limit $N \to \infty$ we can perform a saddle point approximation by expanding the integrand around $k = 0$ i.e. where $\psi(k)$ has its maximum (Gram-Charlier expansion):

$$\frac{\sin(kb)}{kb} = 1 - \frac{(kb)^2}{3!} + \frac{(kb)^4}{5!} + O(k^6) \tag{1.117}$$

and

$$\psi(k) = -\frac{(kb)^2}{6} - \frac{(kb)^4}{180} + O(k^6). \tag{1.118}$$

If we keep just the first term in the expansions we get

$$G(\vec{R}, N) \sim \frac{1}{2\pi^2 R}\int_0^\infty k\sin(kR)e^{-N(kb)^2/6}dk. \tag{1.119}$$

The right hand side is an integral that can be performed and we finally get

$$\boxed{G(\vec{R}, N) \sim \left(\frac{3}{2\pi b^2 N}^{3/2}\right)\exp\left(-\frac{3R^2}{2b^2N}\right).} \tag{1.120}$$

Thus in the $N \to \infty$ limit, the distribution function of the end-to-end vector is Gaussian. To see the first correction to the Gaussian behaviour let us consider also the second term of the expansion. This gives

$$e^{N\psi(k)} = e^{-N(kb)^2/6}e^{-N(kb)^4/180} = e^{-N(kb)^2/6} - \frac{N(kb)^4}{180}e^{-N(kb)^2/6} \tag{1.121}$$

and

$$G(\vec{R}, N) \sim \frac{1}{2\pi^2 R}\left[\int_0^\infty k\sin(kR)e^{-N(kb)^2/6}dk - \frac{Nb^4}{180}\int_0^\infty k^5\sin(kR)e^{-N(kb)^2/6}dk\right]. \tag{1.122}$$

Simplification and integration gives

$$\boxed{G(\vec{R}, N) \sim \left(\frac{3}{2\pi b^2 N}^{3/2}\right)\exp\left(-\frac{3R^2}{2b^2N}\right)\left[1 - \frac{3}{4N} + \frac{3R^2}{2b^2N^2} - \frac{9R^4}{20b^4N^3}\right].} \tag{1.123}$$

The result (1.120) is indeed a consequence of the central limit theorem where the variable S_n corresponds to the end-to-end distance and the i.i.d. variables to the increments $\vec{r}_i$ having $\mathbb{E}\{\vec{r}_i\} = 0$ and $Var\{\vec{r}_i\} = \sigma^2 = b^2$. This gives

$$G(\vec{R}, N) \to \mathbb{N}\left(0, Nb^2\right), \quad \text{as} \quad N \to \infty. \tag{1.124}$$

Note that the above result give the well known property of an ideal chain (random walk)

$$\langle R_e^2\rangle \simeq Nb^2. \tag{1.125}$$

1.5 Martingales

Another particular collection of stochastic process are the so called *Martingales*. We will see that among these processes there are the random walk (discrete in time) and the Brownian motion (continous in time). Martingales play an important role in computations involving stopping times. We first deal with discrete time martingales.

1.5.1 Time discrete martingales

Roughly speaking a discrete time martingale is a sequence of random variables $\{X_0, X_1, \cdots\}$ with finite means (i.e. integrables) and such that the conditional expectation of X_{n+1} given $X_0, X_1, \cdots, X_n$ is equal to X_n. A more general (and slightly more precise) definition of discrete time martingale could be the following:

Definition. A sequence $Y_n = f(X_0, X_1, \cdots, X_n)$ $(n \geq 0)$ is a martingale with respect to the sequence $X_0, X_1, \cdots$ if for all $n \geq 0$

1. $\mathbb{E}\{|Y_n|\} < \infty$, i.e. integrable
2. $E\{Y_{n+1}|X_0, X_1, \cdots, X_n\} = Y_n$

As an example of time discrete martingale let us consider a random walk on the real line starting at $x_0 = 0$ with steps $\{\Delta X_k\}_{k=1,n}$. The k-esim step can in principle depend on its history. Let X_n be the position after n steps i.e. $X_n = \sum_{k=1}^{n} \Delta X_k$. The steps $\{\Delta X_k\}_{k=1,n}$ are the increments of the process X_n. The conditional probability of the kth step given the history $\Delta X_1, \cdots, \Delta X_{k-1}$ is

$$p(\Delta X_k|\Delta X_1, \cdots, \Delta X_{k-1}) = \frac{p(\Delta X_1, \cdots, \Delta X_{k-1}, \Delta X_k)}{p(\Delta X_1, \cdots, \Delta X_{k-1}, \Delta X_{k-1})}. \tag{1.126}$$

The random process $\{X_n\}_{n\geq 0}$ forms a martingale if each increment ΔX_k (or *martingale difference*) has zero conditional mean: i.e. if for each $k \geq 1$

$$\mathbb{E}[\Delta X_k|\Delta X_1, \cdots, \Delta X_{k-1}] = 0 \tag{1.127}$$

for all histories $\{\Delta X_k\}_{k=1,k-1}$. Note that althouh the mean value is independent on the history of the walk, the conditional probability density (1.126) can depend on $\{\Delta X_k\}_{k=1,n}$. If we define the expected value of the location X_n as

$$\mathbb{E}[X_n] = \int x_n p(\Delta X_1, \cdots, \Delta X_n) d\Delta X_1 \cdots \Delta X_n \tag{1.128}$$

it is possible to proof the following Lemma for martingale processes:

Lemma 1.5.1. If $\{X_n\}_{n\geq 0}$ is a martingale with $x_0 = 0$, then

- $\mathbb{E}[X_n] = 0$
- $Var[X_n] = \sum_{k=1}^{n} Var[\Delta X_k]$

This result can be proven inductively by using

$$\int d\Delta x_{n+1} p(\Delta x_{n+1}|\Delta x_1, \cdots, \Delta x_n) = 1 \tag{1.129}$$

and

$$\int d\Delta x_{n+1} \Delta x_{n+1} p(\Delta x_{n+1}|\Delta x_1, \cdots, \Delta x_n) = \mathbb{E}[\Delta X_{n+1}|\Delta X_1, \cdots, \Delta X_n] = 0. \tag{1.130}$$

Before stating the Martingale central limit theorem let us recall that a sequence Y_n of random variables is said to convergence *in probability* to a random variable Y if $\forall \delta > 0$, the probability of $|Y_n - Y| > \delta$ goes to zero as $n \to \infty$.

Theorem 1.5.2 (Martingale central limit theorem)**.** Suppose that $\Delta X_1, \Delta X_2, \cdots$ are square-integrable martingale differences such that

- $$\max_{1\leq k\leq n} \left(|\Delta X_k|\sqrt{n}\right) \to 0 \qquad \text{in probability} \tag{1.131}$$

- $$\sum_{k=1}^{n} \Delta x_k^2/n \to Z^2, \qquad \text{in probability} \tag{1.132}$$

- $$\mathbb{E}\left[\max_{1\le k\le n}(\Delta x_k^2/n)\right] \qquad \text{is bounded in } n, \tag{1.133}$$

where the random variable Z si finite with probability 1; then

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\Delta X_k \to Y, \tag{1.134}$$

where the convergence is in distribution and the random variable Y has characteristic function given by

$$f_Y(k) = \mathbb{E}\left[\exp -\left(\frac{1}{2}z^2k^2\right)\right] \qquad \forall k. \tag{1.135}$$

Observe that the martingale differences ΔX_k are not required to be independent or to be identically distributed.

Lemma 1.5.3. If the random variable Y and Z satisfy eq. (1.135), then Y is Gaussian if and only if Z^2 is a constant.

1.6 Processes with stationary and independent increments

Definition (Process with independent increments)**.** A stochastic process $X(t) \in \mathbb{R}$ has *independent increments* if, $\forall T = \{t_1, t_2, \cdots, t_n\}$

$$X(0), X(t_1) - X(0), X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1}) \tag{1.136}$$

are independent random variables.

Definition (Process with stationary increments)**.** A stochastic process $X(t) \in \mathbb{R}$ has *stationary increments* if $X(t)-X(s)$ has the same probability distribution than $X(t+\tau)-X(s+\tau)$, $\forall s, t, \tau \ge 0$, $s < t$.

If $X(t) \in \mathbb{R}$ is a stochastic process with stationary and independent increments the following properties hold (see Problem)

$$\mathbb{E}\{X(t)\} = \eta_1 t, \tag{1.137}$$

$$Var\{X(t)\} = \sigma_1^2 t \tag{1.138}$$

and

$$\mathbb{E}\{X(t)X(s)\} = \sigma_1^2 \min(t, s) \tag{1.139}$$

where $\eta_1 = \mathbb{E}\{X(1)\}$, $\sigma_1^2 = Var\{X(1)\}$.

1.7 Gaussian processes

In this section we will define the Gaussian stochastic processes and show their basic properties.

Definition (Gaussian process)**.** A Gaussian, zero average, stochastic process $X(t) \in \mathbb{R}$ is a stochastic process in which for any given time partition $T = \{t_1, t_2, \cdots, t_n\}$ the joint probability density function is given by

$$p(x_1, t_1; \ldots x_n, t_n) \equiv \frac{1}{(2\pi)^{n/2}} (\det \mathbf{A})^{1/2} e^{-\frac{1}{2}\sum_{i,j=1}^{n} x_i A_{ij} x_j}, \tag{1.140}$$

where $\mathbf{A} \in M_n(\mathbb{R})$ is a real ($A_{ij} \in \mathbb{R}$), symmetric ($A_{ij} = A_{ji}$), invertible ($\det \mathbf{A} \neq 0$), strictly positive defined ($\sum_{i,j=1}^{n} x_i A_{ij} x_j > 0$), $n \times n$ matrix.

The matrix $\mathbf{A}$ depends of course on the chosen time partition $t_1 \cdots t_n$, but for any given $t_i, i = 1..n$ $\mathbf{A}$ can be diagonalized and all the eigenvalues are strictly positive.

Exercise (Characteristic function of a Gaussian process)**.** Show that the Fourier transform of the n-point joint probability density function of a Gaussian process, $p(x_n, t_n; x_{n-1}, t_{n-1}; \cdots x_1, t_1)$, is given by:

$$f(k_n, t_n; \cdots k_1, t_1) = e^{-\frac{1}{2}\sum_{i,j=1}^{n} k_i \left(A^{-1}\right)_{ij} k_j} \tag{1.141}$$

Lemma 1.7.1 (Correlations of a Gaussian process)**.** The n-point correlation functions of a Gaussian process are given by the relation

$$\mathbb{E}\{X(t_1) \cdots X(t_n)\} = (-i)^n \frac{\partial}{\partial k_1} \cdots \frac{\partial}{\partial k_n} f(k_n, t_n; \cdots k_1, t_1) \mid_{k_i = 0 \forall i} . \tag{1.142}$$

Proof. Just notice that

$$\begin{aligned} \mathbb{E}\{X(t_1) \cdots X(t_n)\} &= \int_{\mathbb{R}^n} dx_1 \cdots dx_n x_1 \cdots x_n p(k_n, t_n; \cdots k_1, t_1) \\ &= (-i)^n \frac{\partial}{\partial k_1} \cdots \frac{\partial}{\partial k_n} \underbrace{\int_{R^n} dx_1 \cdots dx_n e^{\langle \mathbf{k} | \mathbf{x} \rangle} p(x_1, t_1; \cdots x_n, t_1)}_{= f(k_1, t_1; \cdots k_n, t_n)} \mid_{k_i = 0 \forall i} \end{aligned} \tag{1.143}$$

□

Exercise. Show that for a zero-average Gaussian process the two-time correlation (covariance) $C(t_i, t_j)$ is given by:

$$\boxed{C(t_i, t_j) \equiv \mathbb{E}\{X(t_i) X(t_j)\} = \left(\mathbf{A}^{-1}\right)_{ij}.} \tag{1.144}$$

The above result is showing that the covariance determines all the matrix elements of $\mathbf{A}^{-1}$ and thus of $\mathbf{A}$. Hence, a zero-average Gaussian process is completely determined by its covariance. As a consequence, the covariance must also determine all higher-order correlation terms. This result is summarized in the following theorem.

Theorem 1.7.2 (Wick's theorem)**.** For a zero-average Gaussian process we have:

$$\mathbb{E}\{X(t_1) \cdots X(t_n)\} = \begin{cases} \sum_{\mathcal{P}(n)} \mathbb{E}\{X(t_{p_1}) X(t_{p_2})\} \cdots \mathbb{E}\{X(t_{p_{n-1}}) X(t_{p_n})\}, & n \quad \text{even}, \\ 0, & n \quad \text{odd}, \end{cases} \tag{1.145}$$

where the sum is over all the partitions $\mathcal{P}(n)$ of $1, \cdots, n$ in $k = n/2$ elements, with n even. The number of such terms is $(2k-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots (2k-1) = \frac{(2k)!}{k! 2^k}$.

For a general proof see (TODO). With the aid of Eq (1.142) one can easily verify that for the 4-point correlation function we have

$$\begin{aligned} \mathbb{E}\{X(t_1) X(t_2) X(t_3) X(t_4)\} &= \mathbb{E}\{X(t_1) X(t_2)\} \mathbb{E}\{X(t_3) X(t_4)\} + \mathbb{E}\{X(t_1) X(t_3)\} \mathbb{E}\{X(t_2) X(t_4)\} \\ &+ \mathbb{E}\{X(t_1) X(t_4)\} \mathbb{E}\{X(t_2) X(t_3)\} \end{aligned} \tag{1.146}$$

Observation. The correlation functions of the free quantum fields follow the same relations and the above theorem in quantum field theory is known as *Wick's theorem.*

Remark. Note that the result of Eq. (1.144) tell us that a Gaussian process that is uncorrelated it is also completely random (i.e. the $X(t_i)$ are independent $\forall i$). Indeed if the Gaussian process is uncorrelated we have:

$$\mathbb{E}\{X(t_i)X(t_j)\} = \delta_{ij} = \left(\mathbf{A}^{-1}\right)_{ij} \tag{1.147}$$

Hence the matrix $\mathbf{A}$ is the identity and from Eq. (1.140) this means that the joint density distribution is just a product of n Gaussian random variables.

Lemma 1.7.3 (Linear transformation)**.** A stochastic process obtained from a Gaussian process through a linear transformation is still a Gaussian process.

Definition (Stationary Gaussian process)**.** For a general stationary process the two-point correlation function depends only on the time difference:

$$\mathbb{E}\{X(t_i)X(t_j)\} = C(t_j - t_i). \tag{1.148}$$

For a stationary Gaussian process, this function determines the whole process. Hence, a stationary Gaussian process is completely defined by a function $C(t)$ positively defined.

Remark. If a Gaussian process is stationary in a weak sense then it is also stationary in the strong sense (see Problems section). This is somehow understandable since a Gaussian process is completely determined by the first and second moments to which the definition of weak stationarity is applied.

Observation. If, more generally, a Gaussian process has $\mathbb{E}\{X(t)\} \neq 0$, one can verify that the stochastic process $X(t) - \mathbb{E}\{X(t)\}$ is a zero-average Gaussian process, for which all the above results are valid.

Up to now we defined two important classes of stochastic processes: the Markovian and the Gaussian processes. Dobb's theorem characterizes the conditions under which these two classes are equivalent.

Theorem 1.7.4 (Dobb's theorem)**.** A stationary Gaussian process is also a Markovian process if and only if its autocorrelation function $C(t)$ is an exponential function.

Definition (Gaussian process with independent increments)**.** A zero-average Gaussian process has *independent increments* if the random variables

$$X(t_2) - X(t_1), \cdots X(t_n) - X(t_{n-1}) \qquad \text{are independent} \tag{1.149}$$

for all $0 \leq t_1 < \cdots < t_n < \infty$.

Lemma 1.7.5. A zero-average Gaussian process has independent increments if and only if such increments are not correlated, i.e.,

$$\mathbb{E}\{[X(s) - X(r)][X(u) - X(t)]\} = 0 \tag{1.150}$$

for all $0 \leq r < s \leq t < u < \infty$.

Proof. We have previously seen that if two random variables are independent, then they are also not correlated. This is valid in general for random variables, and in particular for normal random variables. Independency is a stronger condition than the absence of correlation and in general two random variables can be not correlated without being independent. In contrast, for normal random variables we have seen that independency and absence of correlation are equivalent properties. This demonstrates the lemma. □

1.8 Wiener-Levy process

An important example of a zero-average Gaussian process with stationary independent increments is the Wiener-Levy process $W(t)$.

Definition (Wiener-Levy process)**.** A stochastic process $\{W(t)\}$, $t \geq 0$ is a Wiener-Levy process if

1. $W(0) = 0$, $W(t)$ is real.
2. $\mathbb{E}\{W(t)\} = 0$.
3. $W(t)$ has **independent increments**.
4. $W(t)$ has **stationary incrementa**, i.e. the probability density function of $W(t) - W(s)$ only depends on $(t - s)$.
5. The increments $W(t) - W(s)$ follow a Gaussian distribution.

Note. In the mathematical community the Wiener process is often called *Brownian motion process*. To avoid confusion with the Brownian motion introduced in chapter 1 the name Wiener process is taken.

From the properties of stochastic process with stationary independent increments (see subsection above) one gets that a Wiener process is characterized by

$$W(0) = 0; \tag{1.151}$$

and by an average and a autocorrelation function

$$\mathbb{E}\{W(t)\} = 0 \tag{1.152}$$

$$\mathbb{E}\{W(t_1), W(t_2)\} = \begin{cases} \sigma^2 t_1 & t_1 \geq t_2 \\ \sigma^2 t_2 & t_1 \leq t_2 \end{cases} = \sigma^2 \min(t_1, t_2), \tag{1.153}$$

with $\sigma \in \mathbb{R}$. It is also easy to show (see Problem) that the Wiener process is a Gaussian process.

Since the Wiener process is Gaussian, the quantities in (1.153) determine uniquely its statistics. For example the single-variable reduced probability density function, $p(w, t)$, is easily obtained once noticed that the process $W(t)$ has zero-average and variance

$$\mathbb{E}\{W(t)W(t)\} = \sigma^2 t. \tag{1.154}$$

As a consequence, the single-variable reduced probability density function is

$$p(w, t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-w^2/2\sigma^2 t\right]. \tag{1.155}$$

Theorem 1.8.1. It possible to show that a Gaussian process $X(t)$ with autocorrelation given by Eq. (1.153) has independent increments.

Proof. Given

$$t_1 > t_2 > t_3, \tag{1.156}$$

we have

$$\begin{aligned} \mathbb{E}\left\{[X(t_1) - X(t_2)]\,[X(t_2) - X(t_3)]\right\} &= \mathbb{E}\{X(t_1)X(t_2)\} - \{X(t_2)X(t_2)\} - \mathbb{E}\{X(t_1)X(t_3)\} \\ &+ \{X(t_2)X(t_3)\} \\ &= \sigma^2 t_2 - \sigma^2 t_2 - \sigma^2 t_3 + \sigma^2 t_3 = 0. \end{aligned} \tag{1.157}$$

Thus the increments

$$X(t_1) - X(t_2) \quad \text{e} \quad X(t_2) - X(t_3) \tag{1.158}$$

are orthogonal random variables. Since they are Gaussian with zero-average, they must be independent. Moreover, these increments are *stationary*, as

$$\mathbb{E}\left\{[X(t_1) - X(t_2)]^2\right\} = \sigma^2 t_1 + \sigma^2 t_2 - 2\sigma^2 t_2 = \sigma^2(t_1 - t_2). \tag{1.159}$$

□

Note. A Wiener-Levy process has stationary increments, but the process itself *is not a stationary process*. In fact

$$\mathbb{E}\{W(t_1)W(t_2)\} = \sigma^2 \min(t_1, t_2), \tag{1.160}$$

which is not a function of the difference $(t_2 - t_1)$ only.

The Wiener-Lévy process is particular case of a family of processes called Lévy processes.

Definition (Lévy process)**.** A Lévy process, is any **continous-time** stochastic process that starts at 0 and has **stationary independent increments**.

In general, the probability distributions of the increments of any Lévy process are infinitely divisible. In other words there is a Lévy process for any given infinitely divisible probability distribution.

1.9 Poisson processes

In addition to the mentioned Wiener process another well know Lévy process is the *Poisson process*. In this respect a possible definition of a Poisson process can be the following:

Definition (Poisson process)**.** A Poisson process is a Lévy process whose increments $X(t_1) - X(t_2)$ follow a Poisson distribution with expected value $\lambda(t_1 - t_2)$ where λ is the *intensity* or *rate* of the process.

The Poisson process is often related to the problem of *arrivals* and *counting processes*.

1.9.1 Arrival process

Let τ a time variable and consider an experiment starting at $\tau = 0$. The events of this experiment are randomly distributed in time. Say that the first is occurring at $\tau = \tau_1$, the second at $\tau = \tau_2$ and so on. The random variable τ_i denotes the time at which the i-esim event occurs and the values t_i are a realization of the random variable τ. Let

$$\zeta_i = \tau_i - \tau_{i-1} \tag{1.161}$$

the time interval between the occurrence of the $(i-1)$esim event and the i-esim one. The sequence of random variables $\{\zeta_i, i \geq 0\}$ are called *inter-event time process*. The time random variable τ_i can be written as

$$\tau_i = \zeta_1 + \zeta_2 + \cdots + \zeta_i \tag{1.162}$$

and describes the time interval between the occurrence of the event and the initial time. For this reason the process $\{\tau_i\}$ it is also known as *Arrivals process*.

1.9.2 Counting processes

A stochastic process $\{X(t), t \geq 0\}$ is called *counting process* if $X(t)$ represents the total number of events occurred in the time interval $(0, t)$. From this definition it is clear that $X(t)$ must satisfy the following properties:

1. $X(t) \geq 0$ and $X(0) = 0$,
2. The realization of $X(t)$ are integer numbers $(X : \Omega \to \mathbb{N})$
3. $X(s) \geq X(t)$ is $s < t$,
4. $X(t) - X(s)$ corresponds to the number of events occured at the time interval (s, t).

1.9.3 Poisson process as a counting process

A very important counting process is the *Poisson process* that we can define again in this respect as follows:

Definition. A counting process $X(t)$ is a Poisson process with *frequency* (or *intensity*) λ if

1. $X(t)$ has independent increments,
2. The number of events in a time interval of length t (increments) follows the Poisson distribution with average λt, i.e. for all $s, t > 0$

$$\mathbb{P}(X(t+s) - X(s) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}, \qquad k = 0, 1, 2, \cdots \tag{1.163}$$

From condition 3 of the definition it turns out that a Poisson process has stationary increments and that

$$\mathbb{E}\{X(t)\} = \lambda t \tag{1.164}$$

and from the result on the variance of a Poisson distribution one has

$$Var\{X(t)\} = \lambda t. \tag{1.165}$$

Hence the average number of events that occur in the unit interval $(0, 1)$ (or in any unit time interval) is λ (velocity).

It is easy to show that (see Exercises section)

$$\mathbb{E}\{X(t)X(s)\} = \lambda \min(t, s) + \lambda^2 ts \tag{1.166}$$

while

$$\mathbb{E}\{(X(t) - \mathbb{E}\{X(t)\})(X(s)\mathbb{E}\{X(s)\})\} = \lambda \min(t, s). \tag{1.167}$$

1.10 Continuous processes

In general a **continuous stochastic process** is a stochastic process that may be said continuous in time or with respect to an index parameter that is a continous variable.

Definition (Continuous with probability one)**.** A stochastic process $\{X(t)\}_{t\geq 0}$ is **continuous with probability one** or **strongly continuous** at t if

$$\mathbb{P}\left\{\omega \in \Omega \,|\, \lim_{s\to t} |X(s,\omega) - X(t,\omega)| = 0\right\} = 1 \tag{1.168}$$

Definition (Continuous in probability)**.** A stochastic process $\{X(t)\}_{t\geq 0}$ is **continuous in probability** at t if $\forall \epsilon > 0$,

$$\lim_{s\to t} \mathbb{P}\left\{\omega \in \Omega \,|\, |X(s,\omega) - X(t,\omega)| \geq \epsilon\right\} = 0. \tag{1.169}$$

Equivalently $\{X(t)\}_{t\geq 0}$ is continuous in probability at t if

$$\lim_{s\to t} \mathbb{E}\left\{\frac{|X(s) - X(t)|}{1 + |X(s) - X(t)|}\right\} = 0. \tag{1.170}$$

The continuity in probability also holds if the covariance $\mathbb{E}\{X(t)X(s)\}$ is continuous over $\mathbb{E}^+ \times \mathbb{E}^+$.

Definition (Continuous in mean-square)**.** A stochastic process $\{X(t)\}_{t\geq 0}$ is **continuous in mean square** at t if $\mathbb{E}\{|X(t)|^2\} < \infty$ and

$$\lim_{s\to t} \mathbb{E}\{|X(s) - X(t)|^2\} = 0 \tag{1.171}$$

Definition (Continuous in distribution)**.** A stochastic process $\{X(t)\}_{t\geq 0}$ is **continuous in distibution** at t if

$$\lim_{s\to t} P_X(x,s) = P_X(x,t) \tag{1.172}$$

for all points x at which $P(x,t)$ is continuous where $P(x,t)$ is the probability distribution of the stochastic processs $\{X(t)\}_{t\geq 0}$.

Note that all the continuity definition above concern the probability distributions, not the sample paths. For sample paths we have the following definition of continuous process:

Definition (Continuous in sample paths or sample continuous)**.** A stochastic process $\{X(t)\}_{t\geq 0}$ is **sample continuous** if $X(t,\omega)$ is continuous in t for $\mathbb{P}$-almost all $\omega \in \Omega$.

1.10.1 Relations between the different definitions of continuity

- A stochastic process $\{X(t)\}_{t\geq 0}$ that is continuous with probability one is also continuous in probability.
- A stochastic process $\{X(t)\}_{t\geq 0}$ that is continuous in mean-square is continuous in probability.
- Continuity with probability one neither rimplies, nor is implied by, continuity in mean square.
- A stochastic process $\{X(t)\}_{t\geq 0}$ that is continuous in probability is continuous in distribution.

1.10.2 Continuity with probability one and continuity in sample paths

Note that continuity with probability one at time t means that $\mathbb{P}\{A(t)\}$ where

$$A(t) = \{\omega \in \Omega \,|\, \lim_{s\to t} |X(s,\omega) - X(t,\omega)| \neq 0\}, \tag{1.173}$$

and it is reasonable to check whether or not this is true for each t. On the othe hand the continuity in sample paths requires that $\mathbb{P}\{A\} = 0$ where

$$A = \bigcup_{t\in T} A(t) \tag{1.174}$$

is an uncountable union of events i.e. it may be not an event itself. If this is the case $\mathbb{P}\{A\}$ is ill defined. Moreover, even if A is an event, $\mathbb{P}\{A\}$ can be stricly positive even if $\mathbb{P}\{A(t)\} = 0\ \forall t \in T$. This is the case, for example, of the telegraph process.

1.11 Ornstein-Uhlenbeck process

A stochastic process $\{X(t)\}_{t\geq 0}$ is an **Ornstein-Uhlenbeck process** or **Gauss-Markov process** if it is *stationary, Gaussian, Markovian* and *continuous in probability.* A fundamental theorem, due to Doob, ensures that $\{X(t)\}_{t\geq 0}$ necesseraly satisfies the following linear stochastic differential equation (see Chapter on Langevin equation)

$$dX(t) = -\gamma(X(t) - \mu)dt + \sigma dW(t) \tag{1.175}$$

where $dW(t) = W(t+\Delta t) - W(t)$ are the increments of a Wiener process $\{W(t)\}_{t\geq 0}$ with unit variance. The parameters γ, μ and σ are constants. The moments of the Ornstein-Uhlenbeck process are

$$\mathbb{E}\{X(t)\} = \mu, \qquad \mathbb{E}\{X(t)X(s)\} = \frac{\sigma^2}{2\gamma} e^{-\gamma|s-t|}, \tag{1.176}$$

in the unconditional (stationary) case and

$$\begin{aligned} \mathbb{E}\{X(t)|X(0)=x_0\} &= \mu + (x_0 - \mathbb{E}\{X(t)\})e^{-\gamma t} \\ E\{X(t)X(s)|X(0)=x_0\} &= \frac{\sigma^2}{2\gamma}\left(e^{-\gamma|s-t|} - e^{-\gamma(s+t)}\right) \end{aligned} \tag{1.177}$$

in the conditional (asymptotically stationary) case. Note that the latter case, for $\mu = x_0 = 0$, $\sigma = 1$ and $\gamma \to 0^+$ the process reduces to the Wiener process. The former case encompasses idealized **white noise** $\{dW(t)\}_{t\geq 0}$ when $\mu = 0$, $\gamma = \sigma$ and $\gamma \to \infty$. Because of the form of $\mathbb{E}\{X(t)X(s)\}$, the Ornstein-Uhlenbeck process is also known as **coloured noise**.

1.11.1 A simple algorithm to generate OU processes

Let N be a large integer and let $w_0, w_1, \cdots, w_N$ be independent random numbers generated from a normal distribution with mean 0 and $Var(w) = \sigma^2/(2\gamma)$. Let us define $x_0 = \mu + w_0$ for the unconditional case and $x_0 = x(0)$ for the conditional case. The discrete version of the process can be obtained recursively as

$$x_k = \mu + e^{-\gamma T/N}(x_{k-1} - \mu) + w_k\sqrt{1 - exp(-2\gamma T/N)} \tag{1.178}$$

for $1 \leq k \leq N$ and where T is the length of the integration interval $[0, T]$. If one interpolates linearly the values $X(kT/N) = x_k$ the desired path is obtained. For ore sophisticated simulation methods refer for example to [10, 11].

1.12 Self-similar processes

Definition (Self-similar process with index H)**.** A real-valued stochastic process $\{X_{Hss}(t)\}_{t\in\mathbb{R}}$ is *self-similar* with *index* $H > 0$ (H-ss) if, for any $a > 0$,

$$\{X_{Hss}(at)\}_{t\in\mathbb{R}} = \left\{a^H X_{Hss}(t)\right\}_{t\in\mathbb{R}} \tag{1.179}$$

where the equality is in the distributional sense. One often talks about "statistical self-similarity" or "self-affinity" and H is called the scaling *Hurst* exponent.

Note. An H-ss process cannot be stationary. However an important subclass of H-ss processes are the ones having *stationary increments*.

Definition. A real valued stochastic process $\{X_{Hss}(t)\}_{t\in\mathbb{R}}$ is H-sssi if it is *self-similar* with index H and has stationary increments.

H-sssi processes are of great interest in application since, as we will see below and later (see Chapter on anomalous diffusion), they give rise to stationary sequences with remarkable features.

1.13 Fractional Brownian Motion

We have seen that a regular Brownian Motion $W_t$is a Gaussian process with $\mathbb{E}\{W_t\} = 0$,$\mathbb{E}\{W_{t_1}W_{t_2}\} = min(t_1, t_2)$ having **independent** and **stationary** increments. If we relax the independence property of the increments we loose the Markov property of the Brownian motion. This is the case, for example, for the *fractional Brownian motion*.

Definition (Fractional Brownian motion)**.** A fractional Brownian motion (fBm) of Hurst parameter $H \in (0, 1)$ is a real centered ($\mathbb{E}\{W_H(t)\} = 0$) process of covariance

$$\mathbb{E}\{W_H(t)W_H(s)\} = \frac{\sigma^2}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right) \qquad (s, t \geq 0) \tag{1.180}$$

where $\sigma^2 = Var(W_H(1))$. It is called *standard* if $\sigma^2 = 1$.

Note. A fractional Brownian motion is a *Gaussian H-sssi* process with $0 < H < 1$.

If we take $t = s$ we get the second moment of the fBM :

$$\mathbb{E}\{W_H(t)^2\} = \sigma^2 |t|^{2H}. \tag{1.181}$$

The value of H determines what kind of fBm is:

- If $H = 1/2$ the process is in fact a regular Brownian motion;
- If $H > 1/2$ the increments of the process are positively correlated;
- If $H < 1/2$ the increments are negatively correlated.
- The case $H = 1$ is excluded because it corresponds merely to a line $W_1(t) = tW_1(1)$ with random slope $W_1(1)$.

Proposition 1.13.1. Suppose that $\{W_H(t)\}_{t\in\mathbb{R}}$

- is a Gaussian process with mean $\mathbb{E}\{W_H(t)\} = 0$ and with $W_H(0) = 0$,
- $\mathbb{E}\{W_H^2(t)\} = \sigma^2|t|^{2H}$
- has stationary increments

then $\{W_H(t)\}_{t\in\mathbb{R}}$ is a fractional Brownian motion.

Representation of fBm

An integral representation of the fBm must be of the general form

$$W_H(t) = \int_{\mathbb{R}} g_t(s) dW(s) \tag{1.182}$$

where $W(s)$ is the Wiener process and $dW(s)$ is the white noise. A possible form of $g_t(s)$ was introduced by Mandelbrot and Van Ness (1968) gives the so called integral representation of the fBm of type I:

$$\begin{aligned} W_H(t) &= \int_{-\infty}^{0} \left\{ (t-s)^{H-1/2} - (0-s)^{H-1/2} \right\} dW(s) \\ &+ \int_0^t (t-s)^{H-1/2} dW(s) \end{aligned} \tag{1.183}$$

Is the process defined above self-similar ? To see that take $a > 0$. Then

$$\begin{aligned} W_H(at) &= \int_{-\infty}^{0} \left\{ (at-as)^{H-1/2} - (0-as)^{H-1/2} \right\} dW(as) \\ &+ \int_0^t (at-as)^{H-1/2} dW(as) = a^H W_H(t) \end{aligned} \tag{1.184}$$

where we have used $dW(as) = a^{1/2} dW(s)$.

Does it have stationary increments ? Yes (TODO)

Is is well defined ? In this case we need to check that $\int_{\mathbb{R}} g_t^2(s) ds < \infty$ TODO.

Alternative representation of fBM

An alternative integral representation can be given as follows: We previously had

$$\begin{aligned} W_H(t) &= \int_{-\infty}^{0} \{(t-s)^{H-1/2} - (0-s)^{H-1/2}\} dW(s) \\ &+ \int_{0}^{t} (t-s)^{H-1/2} dW(s) \\ &= \int_{-\infty}^{\infty} \{(t-s)_+^{H-1/2} - (0-s)_+^{H-1/2}\} dW(s). \end{aligned} \tag{1.185}$$

Let us know consider

$$s_+ = \begin{cases} s & s \geq 0 \\ 0 & s < 0 \end{cases} \tag{1.186}$$

and

$$s_- = \begin{cases} 0 & s > 0 \\ -s & s \leq 0 \end{cases} \tag{1.187}$$

Notice that $s_+ > 0$ and $s_- > 0$. Hence an alternative representation is the following: For any $a \geq 0$, $b \geq 0$ (not both zero):

$$\begin{aligned} W_H(t) &= a \int_{-\infty}^{\infty} \{(t-s)_+^{H-1/2} - (0-s)_+^{H-1/2}\} dW(s) \\ &+ b \int_{-\infty}^{\infty} \{(t-s)_-^{H-1/2} - (0-s)_-^{H-1/2}\} dW(s). \end{aligned} \tag{1.188}$$

Actually for the fBm process there exist lots of equivalent representation.

Some sample path properties of fBm

- For any $0 < H < 1$, the paths of fBm are *continuous* but *not differentiable*.
- The paths of fBm get less weird as H goes from 0 to 1. On this basis, on etypically divides the class of all fBm's into

  Antipersistent $(0 < H < 1/2)$. In this case the increments of fBm tend to have opposite signs and their covariance is negative (see subsection below);

  Chaotic $(H = 1/2)$;

  Persistent $(1/2 < H < 1)$. In this case the covariance of two consecutive increments is positive (smaller ziz-zagging of the paths).

- **Regularity**
  Sample-paths are almost nowhere differentiable. To be precise, almost-all trajectories are Holder continuous of any order strictly less than H, i.e., for each trajectory there exists a constant c such that

$$|W_H(t) - W_H(s)| \leq c|t-s|^{H-\epsilon}, \qquad \forall \epsilon > 0 \tag{1.189}$$

The increment process of the fBm and long-range depedence

Let $\{Z(t)\}_{t\in\mathbb{R}}$ an H-sssi process with $0 < H < 1$ and let

$$X_k = Z(k+1) - Z(k), \qquad k \in \mathbb{Z} \tag{1.190}$$

be their increments.

Definition. If $\{Z(t)\}_{t\in\mathbb{R}}$ is a fBm, then the process of the incremenst $\{X_k\}_{k\in\mathbb{Z}}$ is called the *fractional Gaussian noise* (fGn).

In a continuum version of the increment process we can also write

$$W_H(t) = \int_0^t X(t')dt' \tag{1.191}$$

Proposition 1.13.2. The increment-sequence $\{X_k\}_{k\in\mathbb{Z}}$ has the following properties

(a) $\{X_k\}_{k\in\mathbb{Z}}$ is stationary;

(b) $\mathbb{E}\{X_k\} = 0$ (zero mean) ;

(c) $\mathbb{E}\{X_k^2\} = \sigma^2 = \mathbb{E}\{W_H(1)^2\}$

(d) The autocovariance function of the process $\{X_k\}_{k\in\mathbb{Z}}$ is given by

$$Cov(k) = \mathbb{E}\{X_i X_{i+k}\}\frac{\sigma^2}{2}\left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\right). \tag{1.192}$$

(e) Let $k \neq 0$. Then $Cov(k) = 0$ if $H = 1/2$; $Cov(k) < 0$ is $0 < H < 1/2$ and $Cov(k) > 0$ if $1/2 < H < 1$.

(f) If $H \neq 1/2$ then

$$Cov(k) \sim \sigma^2 H(2H-1)|k|^{2H-2}, \qquad \text{as } k \to \infty. \tag{1.193}$$

Note that in the time continuum version the covariance of the increment process $X(t)$ is given by

$$\mathbb{E}\{X(t_1)X(t_2)\} = \sigma^2 H(2H-1)|t_1 - t_2|^{2H-2} + 2\sigma^2 H|t_1 - t_2|^{2H-1}\delta(t_1 - t_2). \tag{1.194}$$

Suppose also that $Cov(k)$ has a spectral density $f(\nu)$ that is

$$Cov(k) = \int_{-\pi}^{\pi} e^{i\nu k} f(\nu)d\nu, \quad k \in \mathbb{Z}, \tag{1.195}$$

and

$$f(\nu) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} e^{-i\nu k} Cov(k), \qquad \nu \in [-\pi, \pi]. \tag{1.196}$$

Then

$$Cov(k) \sim \sigma^2 H(2H-1)|k|^{2H-2}, \qquad \text{as } k \to \infty. \tag{1.197}$$

$$f(\nu) \sim A\nu^{1-2H}, \qquad \text{as } \nu \to \infty. \tag{1.198}$$

When $1/2 < H < 1$ one says that there is *long-range dependence*, or *long memory* or *1/f noise*. Long range dependence corresponds to a divergence of the spectral density $f(\nu)$ at the origin.

Note. The self-similarity and long memory properties make the fractional Brownian motion a suitable input noise in a variey of models. Recently, for example, fBm has been applied in connection with financial time series, hydrology and telecommunications. In order to develop these applications there is need for a stochastic calculus with respect to the fBm. Nevertheless, fBm is, for $H \neq 1/2$, neither a semimartingale nor a Markov process and, for this reason, the usual stochastic (Ito) calculus cannot be applied to analyze it. It is however possible to define stochastic calculus with respect to fBm (called *fractional stochastic integrals*). Moreover, since the fBm is a Gaussian process, it possible to apply the stochastic calculus of variations which is valid on general Wiener spaces (Ito-Clark formula and Girnasov theorem).

A discrete version of fBm is based on correlated random walks, i.e., discrete processes such that the law of each move is ruled by the value of the previous move. In particular the model one can refer to is the **correlated random walk with persistence p**. It is a process evolving on $\mathbb{Z}$ by jumps of $+1$ or -1, whose probability of making the same jump as the previous one is p (semi-flexible polymer).

Exercise. Let W_t^H be a fractional Brownian motion with Hurst exponent $H \in (0,1)$ and define $b_j^H \equiv W_j^H - W_{j-1}^H$. Calculate $\mathbb{E}(b_j^H\ b_{j+k}^H)$ for $k \gg 1$.

1.14 ARCH Processes

TODO

Exercises

1. Let $X(t) = A\cos(\omega_0 t + \Theta)$ where A and ω_0 are constants and Θ is a continous random variable uniformly distributed on $(-\pi, \pi)$.

(i) Show that $X(t)$ is stationary stochastic process in the weak sense (WSS).

(ii) Compute the spectral density $S(\omega)$:

2. What is the correlation function of the stationary process with spectrum

$$S(\omega) = \frac{1}{\sqrt{\pi|\omega|}} e^{-|\omega|} \qquad ? \tag{1.199}$$

What is the largest-t behaviour of this correlation function ?

3. Show that, for a stochastic process $\{\xi(t)\}$ ($\xi(0) = 0$), with stationary and independent increments, the following properties hold:

(A)

$$\mathbb{E}\{\xi(t)\} = \eta_1 t, \tag{1.200}$$

(B)

$$Var\{\xi(t)\} = \sigma_1^2 t \tag{1.201}$$

(C)

$$Cov\{\xi(t), \xi(s)\} = \mathbb{E}\left\{(\xi(t) - \mathbb{E}\{\xi(t)\})\,(\xi(s) - \mathbb{E}\{\xi(s)\})\right\} = \sigma_1^2 \min(t, s) \tag{1.202}$$

where $\eta_1 = \mathbb{E}\{\xi(1)\}$, $\sigma_1^2 = Var\{\xi(1)\}$.

4. Show that a Gaussian process, stationary in the weak sense, is also stationary in the strong sense. (Hint: work with the characteristic function. Note that we do not need the Gaussian process to have null averages.)

5. Show that for Poisson process with velocity λ the increments are stationary and, by using the independence property of the increments, show that the following properties hold

$$\mathbb{E}\{X(t)X(s)\} = \lambda \min(t, s) + \lambda^2 ts \tag{1.203}$$

and

$$\mathbb{E}\{(X(t) - \mathbb{E}\{X(t)\})(X(s) - \mathbb{E}\{X(s)\})\} = \lambda \min(t, s). \tag{1.204}$$

6. Show that the inter-event times ζ_i (i.e. the time interval between two consecutive events) in a Poisson process with frequency λ are independent and identically distributed exponential random variables with parameter λ. Compute the first moment and the variance of the single point distribution as a function of λ.

7. Let us consider the stochastic process

$$\eta(t) = (-1)^{\xi(t)} \tag{1.205}$$

where $\xi(t)$ is a Poisson process with velocity λ. Hence $\eta(t)$ begins in $\eta(0) = 1$ and it commutes back and forth from $+1$ and -1 at the Poisson random instants T_i. The process $\eta(t)$ is known as *Pseudo-random telegraphic signal* given that the initial value $\xi(t) = 1$ is not random.

(A) Compute the average of $\eta(t)$.

(B) Compute the autocorrelation function of $\eta(t)$.

Let us now consider the random process

$$\zeta(t) = A\eta(t) \tag{1.206}$$

where $\eta(t)$ is the pseudo-random telegraphic process and A is a random variable independent from $\eta(t)$ that can assume the values ± 1 with equal probability. The process is known as *random telegraphic signal*.

(C) Show that $\zeta(t)$ is a stationary process in the weak sense.

(D) Compute the spectrum $S(\omega)$ of the process $\zeta(t)$.

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