

## Moto armonico smorzato

Corpo soggetto ad una forza elastica ed a una forza resistente proporzionale alla velocità :

$$m \vec{a} = \vec{F}_{el} - \lambda \vec{v}$$

$$\rightarrow m \frac{d^2 x(t)}{dt^2} = -k x(t) - \lambda \frac{dx(t)}{dt}$$

$$\frac{d^2 x(t)}{dt^2} + 2\gamma \frac{dx(t)}{dt} + \omega_0^2 x(t) = 0$$

“coefficiente di smorzamento”:  $\gamma \equiv \lambda/2m$

$$\omega_0 \equiv \sqrt{\frac{k}{m}} \quad \text{“pulsazione propria”}$$

Si hanno tre possibili casi:

- |                     |                   |                         |
|---------------------|-------------------|-------------------------|
| $\gamma > \omega_0$ | $\leftrightarrow$ | ”moto sovrasmorzato”    |
| $\gamma = \omega_0$ | $\leftrightarrow$ | ”smorzamento critico”   |
| $\gamma < \omega_0$ | $\leftrightarrow$ | ”oscillazioni smorzate” |

Soluzione di un'equazione differenziale lineare omogenea  
a coefficienti costanti:

$$\frac{d^2 x(t)}{dt^2} + 2\gamma \frac{dx(t)}{dt} + \omega_0^2 x(t) = 0$$

Posto:

$$x(t) \equiv e^{\alpha t}$$

$$\left[ \Rightarrow \frac{dx(t)}{dt} = \alpha e^{\alpha t}, \frac{d^2 x(t)}{dt^2} = \alpha^2 e^{\alpha t} \right]$$

→  $\alpha^2 e^{\alpha t} + 2\gamma \alpha e^{\alpha t} + \omega_0^2 e^{\alpha t} = 0$

→ “**Equazione (algebrica) caratteristica**” associata  
all'equazione differenziale:

$$\alpha^2 + 2\gamma\alpha + \omega_0^2 = 0$$

soluzione:

$$\alpha = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

## Moto “sovramorzato” : $\gamma > \omega_0$

→  $\alpha_1, \alpha_2$  soluzioni **reali** dell'eq.caratteristica;

Soluzione generale:

$$x(t) = A e^{\alpha_1 t} + B e^{\alpha_2 t}$$

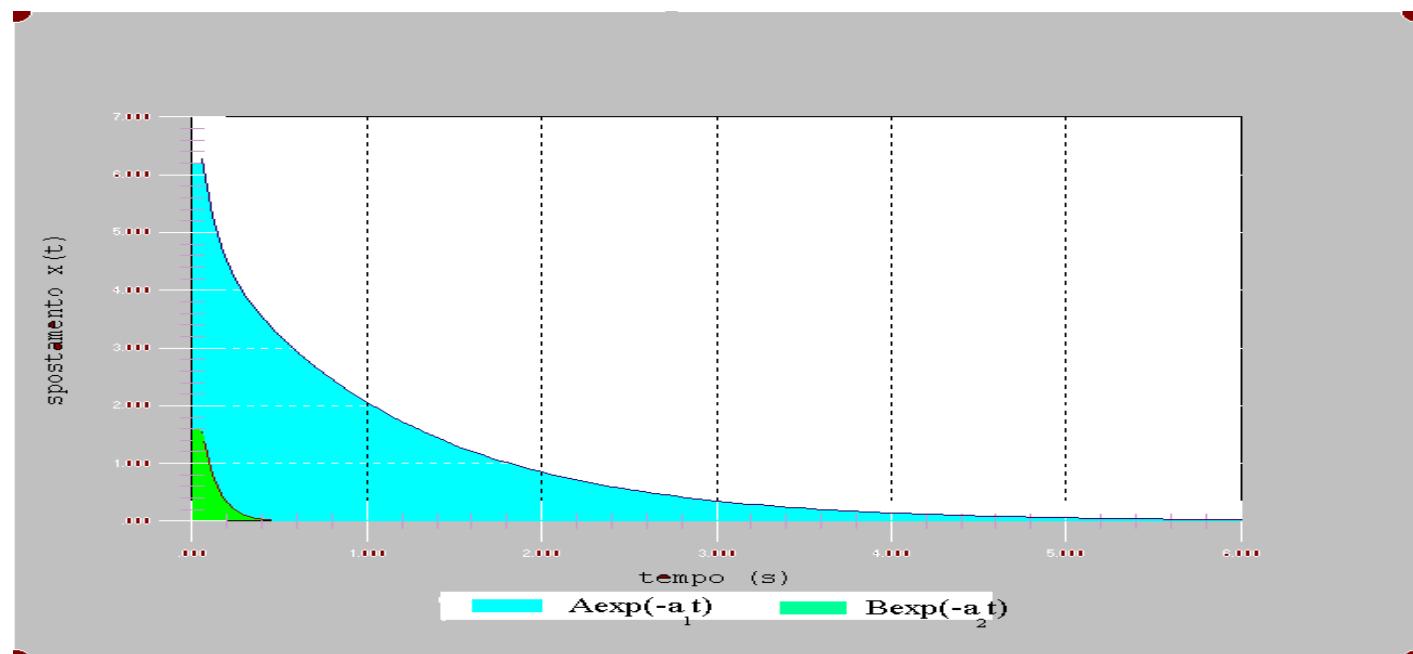
$$= A e^{(-\gamma + \sqrt{\gamma^2 - \omega_0^2})t} + B e^{(-\gamma - \sqrt{\gamma^2 - \omega_0^2})t}$$

$$x(t) = A e^{-(\gamma - \sqrt{\gamma^2 - \omega_0^2})t} + B e^{-(\gamma + \sqrt{\gamma^2 - \omega_0^2})t}$$

Esempio:

$$\omega_0 = 3.14 s^{-1}$$

$$\gamma = 6.00 s^{-1}$$



## “Smorzamento critico”: $\gamma = \omega_0$

$$\rightarrow \frac{d^2 x(t)}{dt^2} + 2\gamma \frac{dx(t)}{dt} + \gamma^2 x(t) = 0$$

$$\rightarrow \frac{d^2 x(t)}{dt^2} + \gamma \frac{dx(t)}{dt} + \gamma \frac{dx(t)}{dt} + \gamma^2 x(t) = 0$$

$$\rightarrow \frac{d}{dt} \left[ \frac{dx(t)}{dt} + \gamma x(t) \right] + \gamma \left[ \frac{dx(t)}{dt} + \gamma x(t) \right] = 0$$

$\equiv z(t)$

$$\rightarrow \frac{dz(t)}{dt} + \gamma z(t) = 0 \quad \rightarrow \quad z(t) = A e^{-\gamma t}$$

Pertanto:

$$\frac{dx}{dt} + \gamma x(t) \equiv z(t) = A e^{-\gamma t}$$

$$\rightarrow e^{\gamma t} \frac{dx}{dt} + \gamma e^{\gamma t} x(t) = A \quad \rightarrow \quad \frac{d \left[ e^{\gamma t} x(t) \right]}{dt} = A$$

$$\rightarrow e^{\gamma t} x(t) = A t + B \quad \rightarrow \quad x(t) = e^{-\gamma t} (A t + B)$$

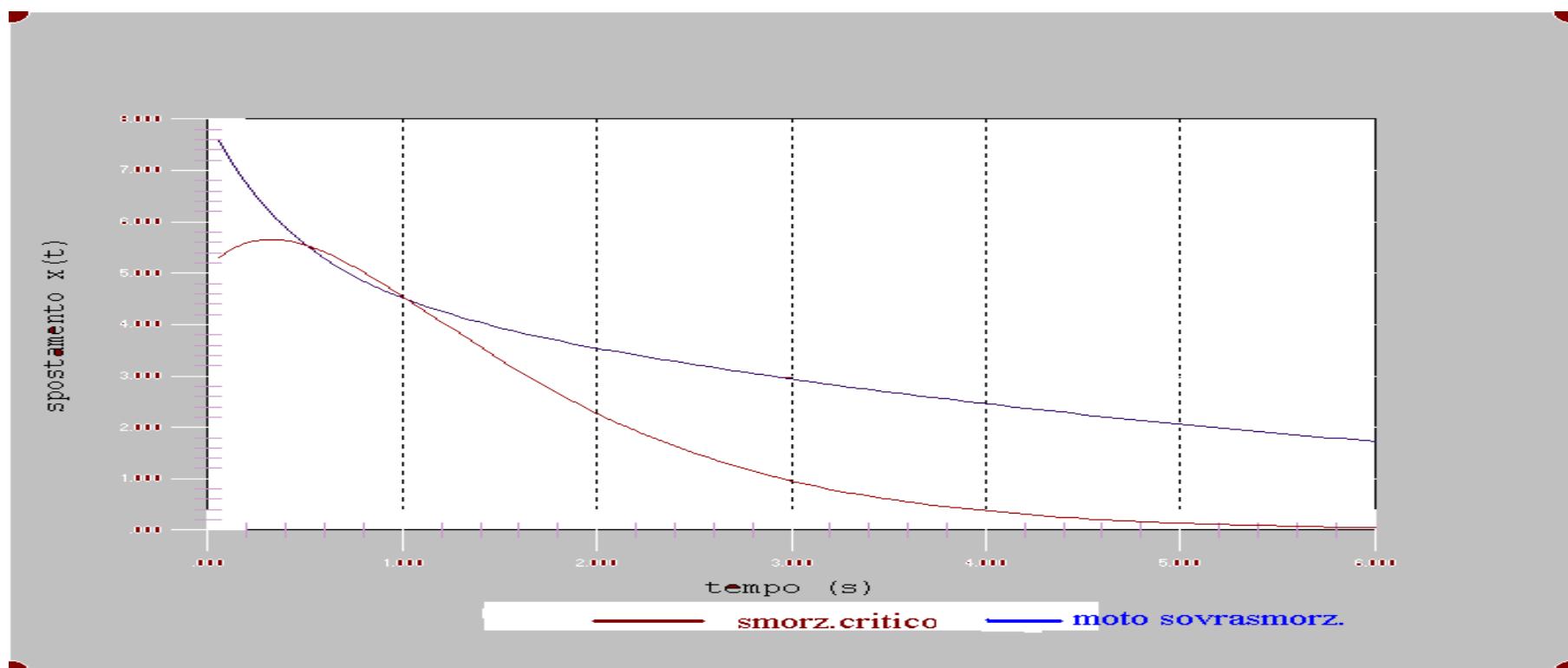
Leggi orarie del moto:

— moto “sovrasmorzato”:

$$x(t) = A e^{-(\gamma - \sqrt{\gamma^2 - \omega_0^2})t} + B e^{-(\gamma + \sqrt{\gamma^2 - \omega_0^2})t}$$

— moto con “smorzamento critico”:

$$x(t) = e^{-\gamma t} (A t + B)$$



## Moto oscillatorio smorzato : $\gamma < \omega_0$

$\alpha_1, \alpha_2$  soluzioni **complesse** dell'eq.caratteristica :

$$\begin{aligned}
 \rightarrow x(t) &= A e^{\alpha_1 t} + B e^{\alpha_2 t} \\
 &= A e^{(-\gamma + i\omega)t} + B e^{(-\gamma - i\omega)t} \\
 &= e^{-\gamma t} (A e^{i\omega t} + B e^{-i\omega t}) \\
 &\quad \uparrow \qquad \qquad \uparrow \qquad \qquad \cos \omega t - i \sin \omega t \\
 &\quad \qquad \qquad \qquad \cos \omega t + i \sin \omega t \\
 &= e^{-\gamma t} [(A + B) \cos \omega t + i(A - B) \sin \omega t]
 \end{aligned}$$

dove:  $\omega \equiv \sqrt{\omega_0^2 - \gamma^2}$

Imponendo che  **$x(t) \equiv$  funzione reale**  $\rightarrow$  A,B complessi  $A = a + i b$   
 (ossia:  $A+B =$  numero reale  $\qquad \qquad$  coniugati :  $B = a - i b$ )  
 $A - B =$  numero immaginario)

Infatti, posto:

$$\begin{aligned}
 A &= a_1 + i b_1 \quad \rightarrow \quad A + B = a_1 + a_2 + i(b_1 + b_2) \equiv r_1 \\
 B &= a_2 + i b_2 \quad \qquad \qquad A - B = a_1 - a_2 + i(b_1 - b_2) \equiv i r_2 \\
 \text{U.Gasparini, Fisica I} \rightarrow & \quad b_1 + b_2 = 0 \quad \rightarrow \quad b_1 = -b_2 \equiv b \quad 6 \\
 & \quad a_1 - a_2 = 0 \quad \qquad \qquad a_1 = a_2 \equiv a
 \end{aligned}$$

## Soluzione per il moto debolmente smorzato:

$$\begin{aligned}
 & A = a + i b \quad \Rightarrow \quad A + B = 2 a \\
 \Rightarrow \quad & B = a - i b \quad \Rightarrow \quad A - B = 2 i b \\
 \Rightarrow \quad & x(t) = e^{-\gamma t} [2 a \cos \omega t + i^2 2 b \sin \omega t] \\
 & = e^{-\gamma t} [2 a \cos \omega t - 2 b \sin \omega t] \\
 \Rightarrow \quad & x(t) = X_0 e^{-\gamma t} \sin(\omega t + \phi)
 \end{aligned}$$

[ con:  $\tan \phi = -a/b$ ,  $X_0 = -2b \sqrt{1 + (a/b)^2}$  ]

$$\begin{aligned}
 \text{infatti: } & x(t) = X_0 e^{-\gamma t} \sin(\omega t + \phi) \\
 & = X_0 e^{-\gamma t} [\sin \omega t \cos \phi + \cos \omega t \sin \phi] \\
 & \equiv e^{-\gamma t} [2 a \cos \omega t - 2 b \sin \omega t]
 \end{aligned}$$

$$\begin{aligned}
 & X_0 \cos \phi = -2b \quad \tan \phi = -(a/b) \\
 \Rightarrow \quad & X_0 \sin \phi = 2a \quad \Rightarrow \quad X_0 \sin \phi = X_0 \frac{\tan^2 \phi}{\sqrt{1 + \tan^2 \phi}} = 2a
 \end{aligned}$$

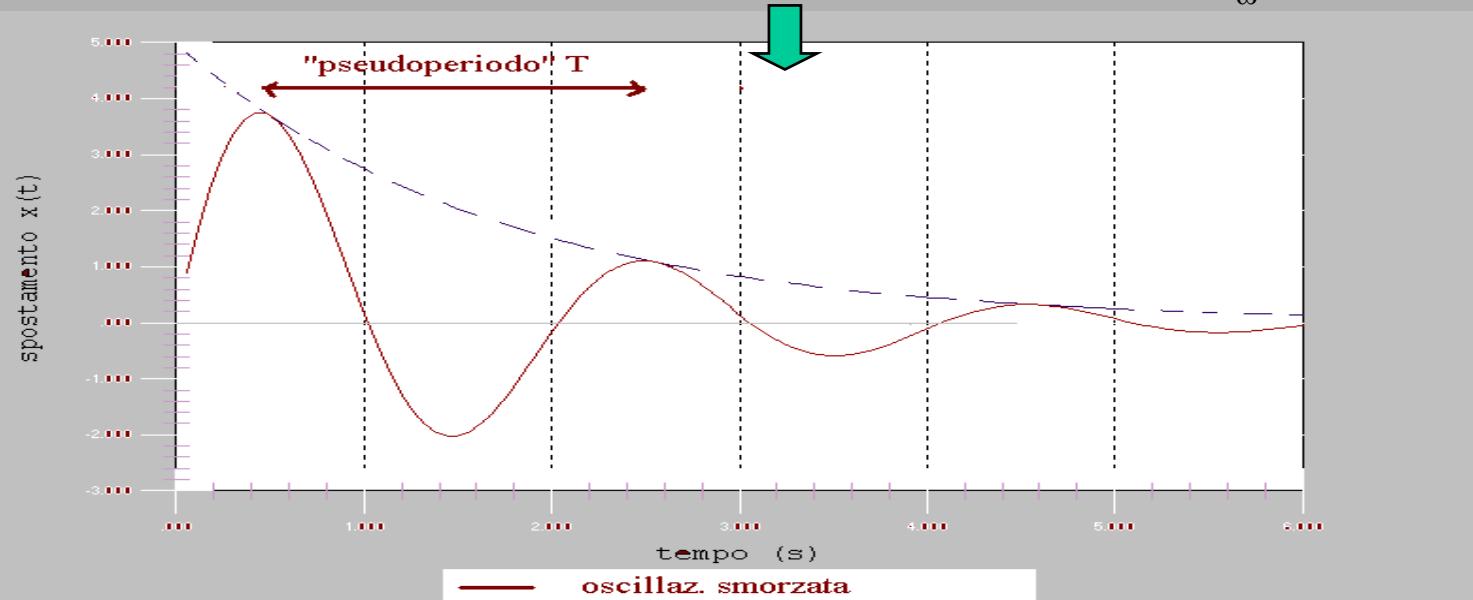
## Soluzione dell'oscillatore armonico con debole smorzamento ( $\gamma < \omega_0$ ): legge oraria

$$x(t) = X_0 e^{-\gamma t} \sin(\omega t + \phi)$$

**“Pseudoperiodo”:**  $T = 2\pi / \omega = \frac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}}$

Esempio:  $\omega_0 = 3.14 \text{ s}^{-1}$ ,  $T_0 \equiv \frac{2\pi}{\omega_0} = 2 \text{ s}$

$$\gamma = 0.6 \text{ s}^{-1} \equiv \omega_0 / 5, \omega \equiv \sqrt{\omega_0^2 - \gamma^2} \equiv 0.97 \omega_0, T \equiv \frac{2\pi}{\omega} \equiv 1.03 T_0$$

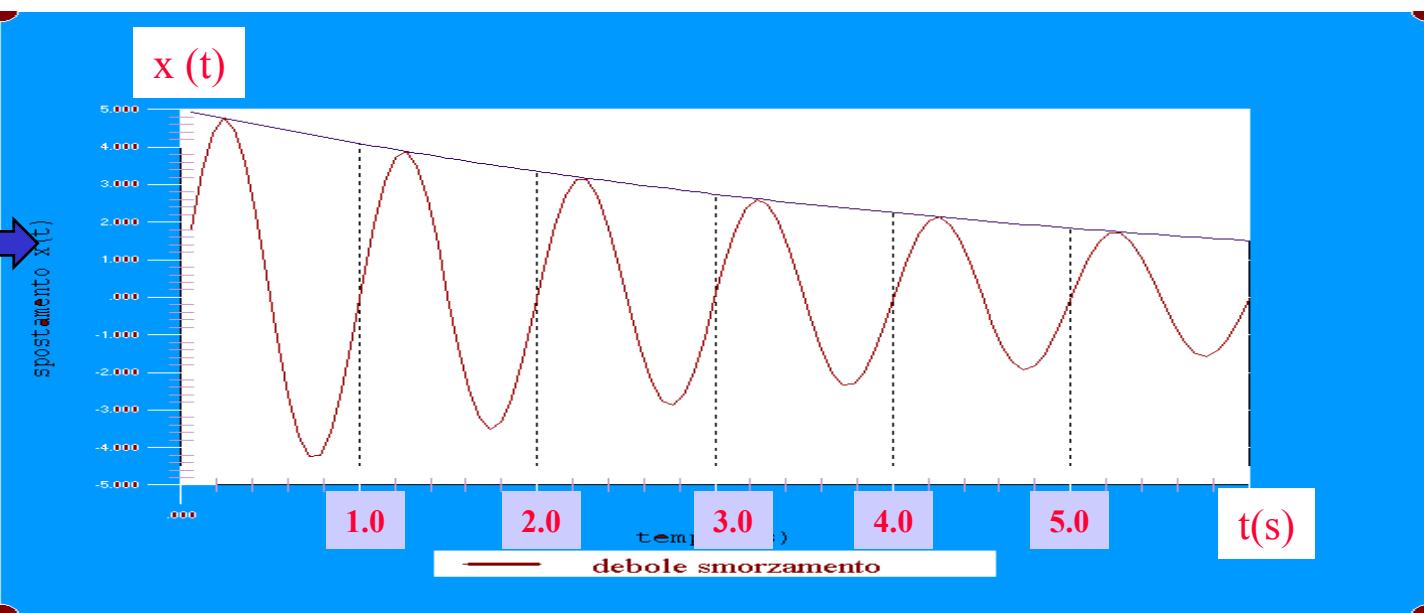


# Esempi:

oscillazione con debole smorzamento:

$$\begin{aligned}\omega_0 &= 6.28 \text{ s}^{-1} \\ T_0 &\equiv \frac{2\pi}{\omega_0} = 1 \text{ s} \\ \gamma &= 0.2 \text{ s}^{-1} \approx \frac{\omega_0}{30} \\ \tau &\equiv \frac{1}{\gamma} = 5 \text{ s} \approx 5T_0\end{aligned}$$

costante di tempo dello smorzamento



$$\begin{aligned}\omega_0 &= 6.28 \text{ s}^{-1} \\ T_0 &\equiv \frac{2\pi}{\omega_0} = 1 \text{ s} \\ \gamma &= 2 \text{ s}^{-1} \approx \frac{\omega_0}{3} \\ \tau &\equiv \frac{1}{\gamma} = 0.5 \text{ s} \approx \frac{T_0}{2}\end{aligned}$$

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oscillazione con forte smorzamento:

