Since the Hamiltonian density has the physical meaning of an *energy density* it could have been computed alternatively, in the Lagrangian formalism, in terms of the canonical energy-momentum tensor associated with the Lagrangian density (10.12). Indeed, from the definition (8.169), and taking into account that we have two independent fields ϕ and ϕ^*, we compute the energy-momentum tensor to be:

$$T_{\mu\nu} = \frac{1}{c}\left[\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi^*)}\partial_\nu\phi^* - \eta_{\mu\nu}\mathcal{L}\right], \tag{10.49}$$

where

$$\frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)} = c^2\partial_\mu\phi^*; \quad \frac{\partial\mathcal{L}}{\partial\partial^\mu\phi^*} = c^2\partial_\mu\phi.$$

Substituting in (10.49) we find:

$$T_{\mu\nu} = c(\partial_\mu\phi^*\partial_\nu\phi + \partial_\nu\phi^*\partial_\mu\phi) - \eta_{\mu\nu}\frac{\mathcal{L}}{c}. \tag{10.50}$$

In particular we may verify the identity between energy density cT_{00} and Hamiltonian density:

$$T_{00} = \frac{1}{c}(2\dot{\phi}^*\dot{\phi} - \mathcal{L}) = \frac{1}{c}\left(\dot{\phi}^*\dot{\phi} + c^2\nabla\phi^*\cdot\nabla\phi + \frac{m^2c^4}{\hbar^2}|\phi|^2\right) = \frac{\mathcal{H}}{c}.$$

that is

$$H = c\int d^3\mathbf{x}T_{00} = \int d^3\mathbf{x}\left(\pi\pi^* + c^2\nabla\phi^*\cdot\nabla\phi + \frac{m^2c^4}{\hbar^2}|\phi|^2\right). \tag{10.51}$$

As far as the momentum of the field is concerned we find

$$P^i = \int d^3\mathbf{x}\left(\dot{\phi}^*\partial^i\phi + \dot{\phi}\partial^i\phi^*\right) \Rightarrow \mathbf{P} = -\int d^3\mathbf{x}\left(\pi\nabla\phi + \pi^*\nabla\phi^*\right). \tag{10.52}$$

10.4 The Dirac Equation

In the previous sections we have focussed our attention on a scalar field, whose distinctive property is the absence of internal degrees of freedom since it belongs to a trivial representation of the Lorentz group. This means that its intrinsic angular momentum, namely its *spin*, is zero.

We have also studied, both at the classical level and in a second quantized setting, the electromagnetic field which, as a four-vector, transforms in the fundamental representation of the Lorentz group. Its internal degrees of freedom are described by the two transverse components of the polarization vector. At the end of Chap. 6 we have associated with the photon a unit spin: $s = 1$ (in units of $\hbar$). As explained there, by this we really mean that the photon *helicity* is $\Gamma = 1$.

Our final purpose is to give an elementary account of the quantum description of electromagnetic interactions. The most important electromagnetic interaction at low energy is the one between matter and radiation. Since the elementary building blocks of matter are electrons and quarks, which have half-integer spin ($s = 1/2$), such processes will involve the interaction between photons and spin 1/2 particles. It is therefore important to complete our analysis of classical fields by including the *fermion* fields, that is fields associated with spin 1/2 particles.

In this section and in the sequel we discuss the relativistic equation describing particles of spin 1/2, known as *the Dirac equation.*

10.4.1 The Wave Equation for Spin 1/2 Particles

Historically Dirac discovered his equation while attempting to construct a relativistic equation which, unlike Klein–Gordon equation, would allow for a consistent interpretation of the modulus squared of the wave function as a probability density. As we shall see in the following, this requirement can be satisfied if, unlike in the Klein–Gordon case, the equation is of *first order in the time derivative*. On the other hand, the requirement of relativistic invariance implies that the equation ought to be of *first order in the space derivatives* as well. The resulting equation will be shown to describe particles of spin $s = \frac{1}{2}$.

Let $\psi^\alpha(x)$ be the classical field representing the wave function. The most general form for a first order wave equation is the following:

$$i\hbar\frac{\partial\psi}{\partial t} = (-ic\hbar\boldsymbol{\alpha}^i\partial_i + \boldsymbol{\beta}mc^2)\psi = \hat{H}\psi. \tag{10.53}$$

In writing (10.53) we have used a matrix notation suppressing the index α of $\psi^\alpha(x)$ and the indices of the matrices $\boldsymbol{\alpha}^i$, $\boldsymbol{\beta}$ acting on ψ^α namely $\boldsymbol{\alpha}^i = (\alpha^i)^\alpha{}_\beta$, $\boldsymbol{\beta} = (\beta)^\alpha{}_\beta$.

In order to determine the matrices $\boldsymbol{\alpha}^i$, $\boldsymbol{\beta}$ we require the solutions to (10.53) to satisfy the following properties:

(i) $\psi^\alpha(x)$ must satisfy the Klein–Gordon equation for a free particle which implements the mass-shell condition:

$$E^2 - |\mathbf{p}|^2c^2 = m^2c^4;$$

(ii) It must be possible to construct a conserved current in terms of ψ^α whose 0-component is *positive definite* and which thus can be interpreted as a probability density;

(iii) Equation (10.53) must be Lorentz covariant. This would imply Poincaré invariance.

To satisfy the first requirement we apply the operator $i\hbar\frac{\partial}{\partial t}$ to both sides of (10.53) obtaining:

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = (-ic\hbar\boldsymbol{\alpha}^i\partial_i + \boldsymbol{\beta}mc^2)(-ic\hbar\boldsymbol{\alpha}^j\partial_j + \boldsymbol{\beta}mc^2)\psi, \tag{10.54}$$

$$\boldsymbol{\alpha}^i\boldsymbol{\alpha}^j\partial_i\partial_j = \frac{1}{2}(\boldsymbol{\alpha}^i\boldsymbol{\alpha}^j + \boldsymbol{\alpha}^j\boldsymbol{\alpha}^i)\partial_i\partial_j, \tag{10.55}$$

where, because of the symmetry of $\partial_i\partial_j$, the term $\boldsymbol{\alpha}^i\boldsymbol{\alpha}^j\partial_i\partial_j$ can be rewritten as If we now require $\boldsymbol{\alpha}^i$ and $\boldsymbol{\beta}$ to be *anticommuting* matrices, namely to satisfy:

$$\left\{\boldsymbol{\alpha}^i, \boldsymbol{\alpha}^j\right\} \equiv \boldsymbol{\alpha}^i\boldsymbol{\alpha}^j + \boldsymbol{\alpha}^j\boldsymbol{\alpha}^i = 2\delta^{ij}\mathbf{1}; \quad \left\{\boldsymbol{\alpha}^i, \boldsymbol{\beta}\right\} = 0, \tag{10.56}$$

and furthermore to square to the identity matrix:

$$\boldsymbol{\beta}^2 = (\boldsymbol{\alpha}^i)^2 = \mathbf{1} \text{ (no summation over } i\text{)}, \tag{10.57}$$

then (10.54) becomes:

$$-\hbar^2\frac{\partial^2\psi^\alpha}{\partial t^2} = (-c^2\hbar^2\nabla^2 + m^2c^4)\psi^\alpha, \tag{10.58}$$

which is the Klein–Gordon equation

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi^\alpha = 0, \tag{10.59}$$

where the differential operator is applied to each component of ψ.

Therefore, given a set of four matrices satisfying (10.56) and (10.57), (10.53) implies the Klein–Gordon equation, as required by our first requirement. Equation (10.53) is called the Dirac equation. We still need to explicitly construct the matrices $\boldsymbol{\alpha}^i$, $\boldsymbol{\beta}$ and to show that requirements (*ii*) and (*iii*) are also satisfied. In order to discuss Lorentz covariance of the Dirac equation, it is convenient to introduce a new set of matrices

$$\gamma^0 \equiv \boldsymbol{\beta}; \quad \gamma^i \equiv \boldsymbol{\beta}\boldsymbol{\alpha}^i, \tag{10.60}$$

in terms of which conditions (10.56) and (10.57) can be recast in the following compact form

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}, \tag{10.61}$$

where, as usual, $i, j = 1, 2, 3$ and $\mu, \nu = 0, 1, 2, 3$. In terms of the matrices γ^μ (10.53) takes the following simpler form[5]:

[5] For the sake of simplicity, we shall often omit the identity matrix when writing combinations of spinorial matrices. We shall for instance write the Dirac equation in the simpler form $\left(i\hbar\gamma^\mu\partial_\mu - mc\right)\psi(x) = 0$.

$$\left(i\hbar\gamma^\mu\partial_\mu - mc\mathbf{1}\right)\psi(x) = 0. \tag{10.62}$$

It can be shown that the minimum dimension for a set of matrices γ^μ satisfying (10.61) is 4. Therefore the simplest choice is to make the internal index α run over four values so that

$$\psi^\alpha(x) = \begin{pmatrix} \psi^1(x) \\ \psi^2(x) \\ \psi^3(x) \\ \psi^4(x) \end{pmatrix} \tag{10.63}$$

belongs to a four-dimensional representation of the Lorentz group.

It must be noted that although the Lorentz group representation $S(\boldsymbol{\Lambda})$ acting on the "vector" ψ has the same dimension as the defining representation $\boldsymbol{\Lambda} = (\Lambda^\mu{}_\nu)$, *the two representations are different.* In our case ψ^α is called a *spinor*, or *Dirac field*, and correspondingly the matrix $S^\alpha{}_\beta$ belongs to the *spinor representation* of the Lorentz group (see next section).[6] This representation will be shown in Sect. 10.4.4 to describe a spin 1/2 particle. This seems to be in contradiction with the fact that ψ has four components, corresponding to its four internal degrees of freedom, which are twice as many as the spin states $s_z = \pm\frac{\hbar}{2}$ of a spin $\frac{1}{2}$ particle. We shall also prove that if we want to extend the invariance from *proper Lorentz transformation* SO(1, 3) to transformations in O(1, 3) which include *parity*, that is including reflections of the three coordinate axes, all the four components of ψ are needed.

It is convenient to introduce an explicit representation of the γ-matrices (10.61), called *standard or Pauli representation*, satisfying (10.61):

$$\gamma^0 = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_2 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \quad (i = 1, 2, 3) \tag{10.64}$$

where each entry is understood as a 2×2 matrix

$$\mathbf{0} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The σ^i matrices are the Pauli matrices of the non-relativistic theory, defined as:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10.65}$$

We recall that they are hermitian and satisfy the relation:

$$\sigma^i\sigma^j = \delta^{ij}\mathbf{1}_2 + i\epsilon^{ijk}\sigma^\kappa, \tag{10.66}$$

which implies

[6] As mentioned in Chap. 7 the spinor representation cannot be obtained in terms of tensor representations of the Lorentz group.

$$\mathrm{Tr}(\sigma^i \sigma^j) = 2\delta^{ij}; \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbf{1}_2; \quad [\sigma^i, \sigma^j] = 2i\epsilon_{ijk}\sigma^k. \tag{10.67}$$

The matrices $\boldsymbol{\alpha}^i$, $\boldsymbol{\beta}$ read:

$$\boldsymbol{\alpha}^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ \sigma^i & \mathbf{0} \end{pmatrix}; \quad \boldsymbol{\beta} = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_2 \end{pmatrix}. \tag{10.68}$$

Using the representation (10.64), the Dirac equation can be written as a coupled system of two equations in the upper and lower components of the Dirac spinor $\psi^\alpha(x)$. Indeed, writing

$$\psi^\alpha(x) = \begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix}; \quad \varphi(x) = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}; \quad \chi(x) = \begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix}, \tag{10.69}$$

where $\varphi(x)$, e $\chi(x)$ are two-component spinors, the Dirac equation (10.62) becomes

$$\left\{ i\hbar c \left[\begin{pmatrix} \mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_2 \end{pmatrix} \frac{\partial}{\partial x^0} + \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix} \frac{\partial}{\partial x^i} \right] - mc^2 \begin{pmatrix} \mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_2 \end{pmatrix} \right\} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 0. \tag{10.70}$$

The matrix equation (10.70) is equivalent to the following system of coupled equations:

$$i\hbar \frac{\partial}{\partial t}\varphi = -i\hbar c \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\chi + mc^2\varphi, \tag{10.71}$$

$$i\hbar \frac{\partial}{\partial t}\chi = -i\hbar c \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\varphi - mc^2\chi, \tag{10.72}$$

where $\boldsymbol{\sigma} \equiv (\sigma^i)$ denotes the vector whose components are the three Pauli matrices. The two-component spinors φ and χ are called *large* and *small* components of the Dirac four-component spinor, since, as we now show, in the non-relativistic limit, χ becomes negligible with respect to φ.

To show this we first redefine the Dirac field as follows:

$$\psi = \psi' e^{-i\frac{mc^2}{\hbar}t}, \tag{10.73}$$

so that (10.62) takes the following form:

$$\left(i\hbar \frac{\partial}{\partial t} + mc^2 \right) \psi' = \left[c\boldsymbol{\alpha}^i \left(-i\hbar\partial_i \right) + \boldsymbol{\beta} mc^2 \right] \psi'.$$

The rescaled spinor ψ' is of particular use when evaluating the non-relativistic limit, since it is defined by "subtracting" from the time evolution of ψ the part due to its rest energy, so that its time evolution is generated by the kinetic energy operator only: $\hat{H} - mc^2\hat{I}$. In other words $i\hbar\partial_t\psi'$ is of the order of the kinetic energy times ψ' and, in the non-relativistic limit, it is negligible compared to $mc^2\psi'$. Next we decompose the field ψ' as in (10.69) and, using (10.68), we find:

$$i\hbar\frac{\partial}{\partial t}\varphi = c\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\chi, \tag{10.74}$$

$$\left(i\hbar\frac{\partial}{\partial t} + 2mc^2\right)\chi = c\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\varphi, \tag{10.75}$$

where we have omitted the prime symbols in the new φ and χ. In the non-relativistic approximation we only keep on the left hand side of the second equation the term $2mc^2\chi$, so that

$$\chi = \frac{1}{2mc}\boldsymbol{\sigma}\cdot\hat{\mathbf{p}}\varphi. \tag{10.76}$$

Substituting this expression in the equation for φ we obtain:

$$i\hbar\frac{\partial}{\partial t}\varphi = \frac{1}{2m}\hat{\mathbf{p}}^2\varphi = -\frac{\hbar^2}{2m}\nabla^2\varphi, \tag{10.77}$$

where we have used the identity:

$$(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma})(\hat{\mathbf{p}}\cdot\boldsymbol{\sigma}) = |\hat{\mathbf{p}}|^2 = -\hbar^2\nabla^2, \tag{10.78}$$

which is an immediate consequence of the properties (10.66) of the Pauli matrices.

Equation (10.77) tells us that in the non-relativistic limit the Dirac equation reduces to the familiar Schroedinger equation for the two component spinor wave function φ. Moreover, from (10.76), we realize that the lower components χ of the Dirac spinor are of subleading order $O(\frac{1}{c})$ with respect to the upper ones φ and therefore vanish in the non-relativistic limit $c \to \infty$. This justifies our referring to them as the *small* and *large* components of ψ, respectively. We also note that in the present non-relativistic approximation, taking into account that the small components χ can be neglected, the probability density $\psi^\dagger\psi = \varphi^\dagger\varphi + \chi^\dagger\chi$ reduces to $\varphi^\dagger\varphi$ as it must be the case for the Schroedinger equation.

10.4.2 Conservation of Probability

We now show that property (*ii*) of Sect. 10.4.1 is satisfied by the solutions to the Dirac equation, namely that it is possible to construct a conserved probability in terms of the spinor ψ^α. Let us take the hermitian conjugate of the Dirac equation (10.62)

$$-i\hbar\partial_\mu\psi^\dagger\gamma^{\mu\dagger} - mc\psi^\dagger = 0. \tag{10.79}$$

We need now the following property of the γ^μ-matrices (10.63) which can be easily verified:

$$\gamma^0 \gamma^{\mu\dagger} = \gamma^\mu \gamma^0. \tag{10.80}$$

Multiplying both sides of (10.79) from the right by the matrix γ^0 and defining the *Dirac conjugate* $\bar{\psi}$ of ψ as

$$\bar{\psi}(x) = \psi^\dagger(x)\gamma^0,$$

we find:

$$-i\hbar\partial_\mu\bar{\psi}\gamma^\mu - mc\bar{\psi} = 0,$$

where we have used (10.80). Thus the field $\bar{\psi}(x)$ satisfies the equation:

$$\bar{\psi}(x)(i\hbar\overleftarrow{\partial}_\mu\gamma^\mu + mc) = 0, \tag{10.81}$$

where, by convention

$$\bar{\psi}\overleftarrow{\partial}_\mu \equiv \partial_\mu\bar{\psi}.$$

Next we define the following *current*:

$$J^\mu = \bar{\psi}\gamma^\mu\psi. \tag{10.82}$$

and *assume* that J^μ transforms as a four-vector. This property will be proven to hold in the next subsection. Using the Dirac equation we can now easily show that $\partial_\mu J^\mu = 0$, that is J^μ *is a conserved current*:

$$\begin{aligned}\partial_\mu J^\mu &= (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi = \bar{\psi}\overleftarrow{\partial}_\mu\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi \\ &= i\frac{mc}{\hbar}\bar{\psi}\psi - i\frac{mc}{\hbar}\bar{\psi}\psi = 0.\end{aligned} \tag{10.83}$$

Note that the 0-component $\rho = J^0 = \psi^\dagger\psi$ of this current is *positive definite*. If we normalize ψ so as to have dimension $[L^{-3/2}]$, then ρ has the dimensions of an inverse volume and therefore it can be consistently given the interpretation of a probability density, the total probability being conserved by virtue of (10.83). The second requirement (*ii*) is therefore satisfied.

10.4.3 Covariance of the Dirac Equation

We finally check that Dirac equation is *covariant* under Lorentz transformations, so that also the third requirement of Sect. 10.4.1 is satisfied

Lorentz covariance of the Dirac equation means that if in a given reference frame (10.62) holds, then in any new reference frame, related to the former one by a Lorentz transformation, the same equation should hold, although in the transformed variables.

Let us write down the Dirac equation in two frames S' and S related by a Lorentz (or in general a Poincaré) transformation:

$$\left(i\hbar\gamma^{\mu}\partial'_{\mu}-mc\right)\psi'(x')=0, \tag{10.84}$$

$$\left(i\hbar\gamma^{\mu}\partial_{\mu}-mc\right)\psi(x)=0, \tag{10.85}$$

where $\partial'_{\mu}=\frac{\partial}{\partial x'^{\mu}}$ and $x'^{\mu}=\Lambda^{\mu}{}_{\nu}x^{\nu}$.

We must require (10.84) to hold in the new frame S' if (10.85) holds in the original frame S.

Let $S\equiv(S^{\alpha}{}_{\beta})=S(\mathbf{\Lambda})$ denote the matrix $\mathbf{D}(\mathbf{\Lambda})=(D^{\alpha}{}_{\beta})$ in (7.47) representing the action of a Lorentz transformation $\mathbf{\Lambda}$ on the spinor components. A Poincaré transformation on $\psi^{\alpha}(x)$ is then described as follows:

$$\psi'_{\alpha}(x')=S^{\alpha}{}_{\beta}\psi^{\beta}(x), \tag{10.86}$$

where, as usual, $x'=\mathbf{\Lambda}x-x_0$. We use a *matrix notation* for the spinor representation while we write explicit indices for the defining representation $\Lambda^{\mu}{}_{\nu}$ of the Lorentz group. Since:

$$\frac{\partial}{\partial x'^{\mu}}=\frac{\partial x^{\nu}}{\partial x'^{\mu}}\frac{\partial}{\partial x^{\nu}}=(\Lambda^{-1})^{\nu}{}_{\mu}\partial_{\nu},$$

we have:

$$\left(i\hbar\gamma^{\mu}\partial'_{\mu}-mc\right)\psi'(x')=\left(i\hbar\gamma^{\mu}(\Lambda^{-1})^{\nu}{}_{\mu}\partial_{\nu}-mc\right)S\psi(x)=0. \tag{10.87}$$

Multiplying both sides from the left by S^{-1} we find:

$$\left[i\hbar(\Lambda^{-1})^{\nu}{}_{\mu}\left(S^{-1}\gamma^{\mu}S\right)\partial_{\nu}-mc\right]\psi(x)=0. \tag{10.88}$$

We see that in order to obtain covariance, we must require

$$(\Lambda^{-1})^{\nu}{}_{\mu}S^{-1}\gamma^{\mu}S=\gamma^{\nu}\Rightarrow S^{-1}\gamma^{\nu}S=\Lambda^{\nu}{}_{\mu}\gamma^{\mu}. \tag{10.89}$$

In that case (10.88) becomes:

$$\left(i\hbar\gamma^{\nu}\partial_{\nu}-mc\right)\psi(x)=0,$$

that is we retrieve (10.85). In the next subsection we shall explicitly construct the transformation S satisfying condition (10.88). We then conclude that *Dirac equation is covariant under Lorentz (Poincaré) transformations*.

We may now check that the current $J^{\mu}=\bar{\psi}\gamma^{\mu}\psi$ introduced in the previous subsection transforms as a four vector. From (10.86) we have, suppressing spinor indices

$$\overline{\psi}'(x') = \overline{S\psi(x)} = \psi^\dagger(x) S^\dagger \gamma^0, \tag{10.90}$$

so that

$$\overline{\psi}'(x')\gamma^\mu \psi'(x') = \psi^\dagger(x)\gamma^0(\gamma^0 S^\dagger \gamma^0)\gamma^\mu S\psi(x) = \overline{\psi}(\gamma^0 S^\dagger \gamma^0)\gamma^\mu S\psi, \tag{10.91}$$

where we have used the property $(\gamma^0)^2 = \mathbf{1}$. As we are going to prove below, the following relation holds:

$$\gamma^0 S^\dagger \gamma^0 = S^{-1}. \tag{10.92}$$

In this case, using (10.89), (10.91) becomes

$$\overline{\psi}'(x')\gamma^\mu \psi'(x') = \overline{\psi}(x) S^{-1}\gamma^\mu S\psi = \Lambda^\mu{}_\nu \overline{\psi}(x)\gamma^\nu \psi(x), \tag{10.93}$$

which shows that *the current J^μ transforms as a four-vector*.

10.4.4 Infinitesimal Generators and Angular Momentum

To find the explicit form of the spinor matrix $S(\mathbf{\Lambda})$ we require it to induce the transformation of the γ-matrices given by (10.89). Actually it is sufficient to perform the computation in the case of infinitesimal Lorentz transformations.

We can write the Poincaré-transformed spinor $\psi'(x')$ in (10.86) as resulting from the action of a differential operator $O_{(\mathbf{\Lambda},x_0)}$, defined in (9.101):

$$\psi'^\alpha(x') = O_{(\mathbf{\Lambda},x_0)}\psi^\alpha(x') = S^\alpha{}_\beta \psi^\beta(x), \tag{10.94}$$

The generators $\hat{J}^{\rho\sigma}$ of $O_{(\mathbf{\Lambda},x_0)}$ are expressed, see (9.102), as the sum of a differential operator $\hat{M}^{\rho\sigma}$ acting on the functional form of the field, and a matrix $\Sigma^{\rho\sigma}$ acting on the internal index α (which coincide with $(-i\hbar)$ times the matrices $(L^{\rho\sigma})^\alpha_\beta$ in (7.83)). These latter are the Lorentz generators in the spinor representation:

$$S(\mathbf{\Lambda}) = e^{\frac{i}{2\hbar}\theta_{\rho\sigma}\Sigma^{\rho\sigma}}, \tag{10.95}$$

and satisfy the commutation relations (9.103):

$$\left[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}\right] = -i\hbar\left(\eta^{\nu\rho}\Sigma^{\mu\sigma} + \eta^{\mu\sigma}\Sigma^{\nu\rho} - \eta^{\mu\rho}\Sigma^{\nu\sigma} - \eta^{\nu\sigma}\Sigma^{\mu\rho}\right). \tag{10.96}$$

We can construct such matrices in terms of the γ^μ ones as follows:

$$\Sigma^{\mu\nu} = -\frac{\hbar}{2}\sigma^{\mu\nu}, \tag{10.97}$$

where the $\sigma^{\mu\nu}$ matrices are defined as:

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu] = -\sigma^{\nu\mu}. \tag{10.98}$$

Using the properties (10.61) of the γ^μ-matrices, the reader can verify that $\Sigma^{\mu\nu}$ defined in (10.97) satisfy the relations (10.96). The expression of an infinitesimal Lorentz transformation on $\psi(x)$ follows from (7.83), with the identification $(L^{\rho\sigma})^\alpha{}_\beta = \frac{i}{\hbar}(\Sigma^{\rho\sigma})^\alpha{}_\beta = -\frac{i}{2}(\sigma^{\rho\sigma})^\alpha{}_\beta$:

$$\begin{aligned}\delta\psi(x) &= \frac{i}{2\hbar}\delta\theta_{\rho\sigma}\hat{J}^{\rho\sigma}\psi(x)\\ &= \frac{1}{2}\delta\theta_{\rho\sigma}\left[-\frac{i}{2}\sigma^{\rho\sigma} + x^\rho\partial^\sigma - x^\sigma\partial^\rho\right]\psi(x),\end{aligned} \tag{10.99}$$

where we have adopted the matrix notation for the spinor indices and used the identification:

$$\hat{J}_{\rho\sigma} = \hat{M}_{\rho\sigma} + \Sigma_{\rho\sigma} = -i\hbar(x_\rho\partial_\sigma - x_\sigma\partial_\rho) - \frac{\hbar}{2}\sigma_{\rho\sigma}. \tag{10.100}$$

To verify that the matrices $\Sigma^{\rho\sigma}$ defined in (10.97) generate the correct transformation property (10.89) of the γ^μ matrices, let us verify (10.89) for infinitesimal Lorentz transformations:

$$\begin{aligned}\Lambda^\mu{}_\nu &\approx \delta^\mu_\nu + \frac{1}{2}\delta\theta_{\rho\sigma}(L^{\rho\sigma})^\mu{}_\nu = \delta^\mu_\nu + \delta\theta^\mu{}_\nu,\\ S(\boldsymbol{\Lambda}) &\approx \mathbf{1} - \frac{i}{4}\delta\theta_{\rho\sigma}\sigma^{\rho\sigma},\end{aligned} \tag{10.101}$$

where we have used the matrix form (4.170) of the Lorentz generators $\mathbf{L}^{\rho\sigma} = [(L^{\rho\sigma})^\mu{}_\nu]$ in the fundamental representation: $(L^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu$. Equation (10.89) reads to lowest order in $\delta\theta$:

$$\left(\mathbf{1} + \frac{i}{4}\delta\theta_{\rho\sigma}\sigma^{\rho\sigma}\right)\gamma^\mu\left(\mathbf{1} - \frac{i}{4}\delta\theta_{\rho\sigma}\sigma^{\rho\sigma}\right) = \gamma^\mu + \frac{1}{2}\delta\theta_{\rho\sigma}(L^{\rho\sigma})^\mu{}_\nu\gamma^\nu.$$

The above equation implies:

$$\frac{i}{2}[\sigma^{\rho\sigma}, \gamma^\mu] = (L^{\rho\sigma})^\mu{}_\nu\gamma^\nu = \eta^{\rho\mu}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho, \tag{10.102}$$

which can be verified using the properties of the γ^μ-matrices. Having checked (10.89) for infinitesimal transformations, the equality extends to finite transformations as well, since the latter can be expressed as a sequence of infinitely many infinitesimal transformations.

As far as (10.92) is concerned, from the definition (10.97) we can easily prove the following property:

$$\gamma^0(\Sigma^{\rho\sigma})^\dagger\gamma^0 = \frac{i\hbar}{4}\gamma^0[\gamma^\rho,\gamma^\sigma]^\dagger\gamma^0 = \frac{i\hbar}{4}\gamma^0[(\gamma^\sigma)^\dagger,(\gamma^\rho)^\dagger]\gamma^0$$
$$= \frac{i\hbar}{4}[\gamma^0(\gamma^\sigma)^\dagger\gamma^0,\gamma^0(\gamma^\rho)^\dagger\gamma^0] = -\frac{i\hbar}{4}[\gamma^\rho,\gamma^\sigma] = \Sigma^{\rho\sigma}.$$

Let us now compute the left hand side of (10.92) by writing the series expansion of the exponential and use the above property of $\Sigma^{\mu\nu}$:

$$\gamma^0 S^\dagger\gamma^0 = \gamma^0\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{2\hbar}\theta_{\rho\sigma}\Sigma^{\rho\sigma\dagger}\right)^n\right]\gamma^0 = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{2\hbar}\theta_{\rho\sigma}\gamma^0\Sigma^{\rho\sigma\dagger}\gamma^0\right)^n$$
$$= \exp\left(-\frac{i}{2\hbar}\theta_{\rho\sigma}\gamma^0\Sigma^{\rho\sigma\dagger}\gamma^0\right) = \exp\left(-\frac{i}{2\hbar}\theta_{\rho\sigma}\Sigma^{\rho\sigma}\right) = S^{-1}. \tag{10.103}$$

This proves (10.92).

In terms of the generators $\hat{J}^{\rho\sigma}$ of the Lorentz group we can define the angular momentum operator $\hat{\mathbf{J}} = (\hat{J}_i)$ as in (9.106) of last chapter:

$$\hat{J}_i = -\frac{1}{2}\epsilon_{ijk}\hat{J}^{jk} = \hat{M}_i + \Sigma_i,$$
$$\hat{M}_i = \epsilon_{ijk}\hat{x}^i\hat{p}^j; \quad \Sigma_i = -\frac{1}{2}\epsilon_{ijk}\Sigma^{jk}, \tag{10.104}$$

where, as usual $\hat{\mathbf{M}} = (\hat{M}_i)$ denotes the orbital angular momentum, while we have denoted by $\boldsymbol{\Sigma} = (\Sigma_i)$ the spin operators acting as matrices on the internal spinor components. Let us compute the latter using the definition (10.97) of $\Sigma^{\mu\nu}$:

$$\Sigma_i = -\frac{1}{2}\epsilon_{ijk}\Sigma^{jk} = \frac{\hbar}{4}\epsilon_{ijk}\sigma^{jk} = \frac{\hbar}{2}\begin{pmatrix}\sigma^i & \mathbf{0}\\ \mathbf{0} & \sigma^i\end{pmatrix}, \tag{10.105}$$

The above expression is easily derived from the definition of σ^{ij} and the explicit form of the γ^μ-matrices:

$$\sigma^{ij} = \frac{i}{2}[\gamma^i,\gamma^j] = -\frac{i}{2}\begin{pmatrix}[\sigma^i,\sigma^j] & \mathbf{0}\\ \mathbf{0} & [\sigma^i,\sigma^j]\end{pmatrix} = \epsilon^{ijk}\begin{pmatrix}\sigma^k & \mathbf{0}\\ \mathbf{0} & \sigma^k\end{pmatrix},$$

where we have used the properties (10.67) of the Pauli matrices and the relation $\epsilon_{ijk}\epsilon^{jk\ell} = 2\delta_i^\ell$. For a massive fermion, like the electron, $\boldsymbol{\Sigma} = (\Sigma_i)$ generate the spin group $G^{(0)} = \mathrm{SU}(2)$, see Sect. 9.4.1, which is the little group of the four-momentum in the rest frame $\mathcal{S}_0$ in which $p = \bar{p} = (mc, \mathbf{0})$. In Sect. 9.4.2 we have shown that

$|\boldsymbol{\Sigma}|^2 = -\hat{W}_\mu \hat{W}^\mu/(m^2c^2)$, i.e. the spin of the particle, is a Poincaré invariant quantity. In our case, using (10.105), we have:

$$|\boldsymbol{\Sigma}|^2 = \hbar^2 s(s+1)\mathbf{1} = \frac{3}{4}\hbar^2\mathbf{1}, \tag{10.106}$$

from which we deduce that the particle has spin $s = 1/2$, namely that the states $|p, r\rangle$ belong to the two-dimensional representation of SU(2), labeled by r. The matrix $\mathcal{R}(\boldsymbol{\Lambda}, p)$ in (9.112) is thus an SU(2) transformation generated by the matrices $\mathbf{s}_i \equiv \hbar\sigma_i/2$, see Appendix F:

$$\mathcal{R}(\boldsymbol{\Lambda}, p) = \exp\left(\frac{i}{\hbar}\theta^i \mathbf{s}_i\right), \tag{10.107}$$

where, if $\boldsymbol{\Lambda}$ were a rotation, θ^i would coincide with the rotation angles, and thus be independent of p, whereas if $\boldsymbol{\Lambda}$ were a boost, θ^i would depend on p and on the boost parameters.

Note that, in the spinorial representation of the Lorentz group, which acts on the index α of $\psi^\alpha(x)$, a generic rotation with angles θ^i is generated by the matrices Σ_i in (10.105) and has the form:

$$S(\boldsymbol{\Lambda}_R) = e^{\frac{i}{\hbar}\theta^i\Sigma_i} = \begin{pmatrix} e^{\frac{i}{\hbar}\theta^i\mathbf{s}_i} & \mathbf{0} \\ \mathbf{0} & e^{\frac{i}{\hbar}\theta^i\mathbf{s}_i} \end{pmatrix} = \begin{pmatrix} \mathbf{S}(\boldsymbol{\theta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{S}(\boldsymbol{\theta}) \end{pmatrix}, \tag{10.108}$$

$$\mathbf{S}(\boldsymbol{\theta}) \equiv e^{\frac{i}{\hbar}\theta^i\mathbf{s}_i} = \cos\left(\frac{\theta}{2}\right) + i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\theta}}\sin\left(\frac{\theta}{2}\right), \tag{10.109}$$

where $\boldsymbol{\theta} \equiv (\theta^i)$, $\theta \equiv |\boldsymbol{\theta}|$ and $\hat{\boldsymbol{\theta}} \equiv \boldsymbol{\theta}/\theta$. Equation (10.109) is readily obtained along the same lines as in the derivation of the 4×4 matrix representation of a Lorentz boost in Chap. 4. Equation (10.108) shows that, with respect to the spin group SU(2), the spinorial representation is completely reducible into two two-dimensional representations acting on the small and large components of the spinor, respectively. Moreover we see that a rotation by an angle θ of the RF about an axis, amounts to a rotation by an angle $\theta/2$ of a spinor.

If the particle is massless, $\mathcal{R}$ is an SO(2) rotation generated by the helicity operator Γ in the frame in which the momentum is the standard one $p = \bar{p}$. Choosing[7] $\bar{p} = (E, 0, 0, E)/c$, $\hat{\Gamma} = \Sigma_3$ and

$$\mathcal{R}(\boldsymbol{\Lambda}, p) = \exp\left(\frac{i}{\hbar}\theta\mathbf{s}_3\right), \tag{10.110}$$

Finally we may verify that the spin $\boldsymbol{\Sigma}$ *does not commute with the Hamiltonian*, i.e. it is not a conserved quantity. Indeed, the expression of the Hamiltonian given in (10.53), namely

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[7] Note that, with respect to the last chapter, we have changed our convention for the standard momentum of a massless particle. Clearly the discussion in Chap. 9 equally applies to this new choice, upon replacing direction 1 with direction 3.

$$H = -ic\hbar\boldsymbol{\alpha}^i\partial_i + \boldsymbol{\beta}mc^2 = c\boldsymbol{\alpha}^i\hat{p}_i + \boldsymbol{\beta}mc^2 = \begin{pmatrix} mc^2 & c\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} \\ c\hat{\mathbf{p}}\cdot\boldsymbol{\sigma} & -mc^2 \end{pmatrix},$$

where we have used the explicit matrix representation (10.68) of $\boldsymbol{\alpha}^i$, $\boldsymbol{\beta}$. Using for $\boldsymbol{\Sigma}$ the expression (10.105) we find:

$$[H, \Sigma^k] = ic\hbar\begin{pmatrix} \mathbf{0} & \epsilon_{kij}\sigma^i\hat{p}^j \\ \epsilon_{kij}\sigma^i\hat{p}^j & \mathbf{0} \end{pmatrix} = ic\hbar\epsilon_{kij}\boldsymbol{\alpha}^j\hat{p}^i \neq \mathbf{0}. \tag{10.111}$$

We see that, considering the third component Σ_3, the commutator does not vanish , except in the special case $p^1 = p^2 = 0$, $p^3 \neq 0$. In general the component of $\boldsymbol{\Sigma}$ along the direction of motion, which is the helicity Γ, is conserved. This is easily proven by computing $[H, \boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}] = [H, \Sigma_i\hat{p}^i]$ and using the property that H commutes with $\hat{p}^i$, so that, in virtue of (10.111), $[H, \Sigma_i\hat{p}^i] = [H, \Sigma_i]\hat{p}^i = 0$.

Similarly also the orbital angular momentum is not conserved since, if we compute $[H, \hat{M}_k]$ and use the commutation relation $[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j$, we find:

$$[H, \hat{M}_k] = \epsilon_{kij}[H, \hat{x}^i]\hat{p}^j = c\epsilon_{kij}\boldsymbol{\alpha}^\ell[\hat{p}_\ell, \hat{x}^i]\hat{p}^j = -ic\hbar\epsilon_{kij}\boldsymbol{\alpha}^i\hat{p}^j.$$

Summing the above equation with (10.111) we find:

$$[H, \hat{J}_k] = [H, \hat{M}_k + \Sigma_k] = -ic\hbar\epsilon_{kij}\boldsymbol{\alpha}^i\hat{p}^j + ic\hbar\epsilon_{kij}\boldsymbol{\alpha}^i\hat{p}^j = \mathbf{0},$$

namely that *the total angular momentum* $\mathbf{J} = \mathbf{M} + \boldsymbol{\Sigma}$ *is conserved.*

So far we have been considering the action of the rotation subgroup of the Lorentz group on spinors. We have learned in Chap. 4 that a generic proper Lorentz transformation can be written as the product of a boost and a rotation:

$$S(\boldsymbol{\Lambda}) = S(\boldsymbol{\Lambda}_B)S(\boldsymbol{\Lambda}_R). \tag{10.112}$$

Let us consider now the boost part. Lorentz boosts are generated, in the fundamental representation, by the matrices $\mathbf{K}_i$ defined in Sect. 4.7.1 of Chap. 4. To find the representation of these generators on the spinors, let us expand a generic Lorentz generator in the spinor representation:

$$\frac{i}{2\hbar}\theta_{\mu\nu}\Sigma^{\mu\nu} = \frac{i}{\hbar}\theta_{0i}\Sigma^{0i} + \frac{i}{\hbar}\theta_i\Sigma^i = \lambda_i K^i + \frac{i}{\hbar}\theta_i\Sigma^i, \tag{10.113}$$

where, as usual, $\theta_i = -\epsilon_{ijk}\theta^{jk}/2$ while $\lambda_i \equiv \theta_{0i}$. The boost generators $K^i = i\Sigma^{0i}/\hbar$ read:

$$K^i = \frac{1}{2}\gamma^0\gamma^i = \frac{1}{2}\boldsymbol{\alpha}^i. \tag{10.114}$$

A boost transformation is thus implemented on a spinor by the following matrix

$$S(\boldsymbol{\Lambda}_B) = e^{\frac{i}{\hbar}\lambda_i\Sigma^{0i}} = e^{\lambda_i K^i}. \tag{10.115}$$

The above matrix can be evaluated by noting that $(\lambda_i K^i)^2 = -\lambda_i\lambda_j\gamma^i\gamma^j/4 = \lambda^2/4$, where $\lambda = |\boldsymbol{\lambda}|$ and we have used the anticommutation properties of the γ^i-matrices. By using this property and defining the unit vector $\hat{\lambda}^i = \lambda^i/\lambda$ the expansion of the exponential on the right hand side of (10.115) boils down to:

$$S(\boldsymbol{\Lambda}_B) = \cosh\left(\frac{\lambda}{2}\right)\mathbf{1} + \sinh\left(\frac{\lambda}{2}\right)\hat{\lambda}^i\boldsymbol{\alpha}_i. \tag{10.116}$$

From the identifications $\cosh(\lambda) = \gamma(v)$, $\sinh(\lambda) = \gamma(v)v/c$, $\hat{\boldsymbol{\lambda}} = (\hat{\lambda}_i) = \mathbf{v}/v$, see Sect. 4.7.1 of Chap. 4, we derive:

$$\cosh\left(\frac{\lambda}{2}\right) = \sqrt{\frac{\gamma(v)+1}{2}};\quad \sinh\left(\frac{\lambda}{2}\right) = \sqrt{\frac{\gamma(v)-1}{2}},$$
$$S(\boldsymbol{\Lambda}_B) = \sqrt{\frac{\gamma(v)+1}{2}}\mathbf{1} + \sqrt{\frac{\gamma(v)-1}{2}}\frac{v^i}{v}\boldsymbol{\alpha}_i. \tag{10.117}$$

It is useful to express the boost $\boldsymbol{\Lambda}_p$ which connects the rest frame $\mathcal{S}_0$ of a massive particle to a generic one in which $p = (p^\mu) = (E_{\mathbf{p}}/c, \mathbf{p})$. In this case we can write $\gamma(v) = E/(mc^2)$, $\mathbf{v}/c = \mathbf{p}c/E_{\mathbf{p}}$ and (10.117), after some algebra, becomes:

$$\begin{aligned} S(\boldsymbol{\Lambda}_p) &= \frac{1}{\sqrt{2m(mc^2+E_{\mathbf{p}})}}(p_\mu\gamma^\mu + mc\gamma^0)\gamma^0 \\ &= \frac{1}{\sqrt{2m(mc^2+E_{\mathbf{p}})}}\begin{pmatrix} (p^0+mc)\mathbf{1}_2 & \mathbf{p}\cdot\boldsymbol{\sigma} \\ \mathbf{p}\cdot\boldsymbol{\sigma} & (p^0+mc)\mathbf{1}_2 \end{pmatrix}. \end{aligned} \tag{10.118}$$

10.5 Lagrangian and Hamiltonian Formalism

The field equations of the Dirac field can be derived from the Lagrangian density:

$$\mathcal{L} = i\frac{\hbar c}{2}\left(\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - \partial_\mu\bar{\psi}(x)\gamma^\mu\psi(x)\right) - mc^2\bar{\psi}(x)\psi(x). \tag{10.119}$$

Indeed, since

$$\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\psi}(x)} = -i\frac{\hbar c}{2}\gamma^\mu\psi(x),$$

we find

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}(x)} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\partial_\mu\bar{\psi}(x)}\right) = 0 \Leftrightarrow \left(i\hbar\gamma^\mu\partial_\mu - mc\mathbf{1}\right)\psi(x) = 0, \tag{10.120}$$

that is, the Dirac equation.

In an analogous way we find the equation for the Dirac conjugate spinor $\bar{\psi}(x)$:

$$\frac{\partial \mathcal{L}}{\partial \psi}(x) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \psi(x)} \right) = 0 \Leftrightarrow \bar{\psi}(x) \left(i\hbar\gamma^\mu \overleftarrow{\partial}_\mu + mc\mathbf{1} \right) = 0. \tag{10.121}$$

We note that the Lagrangian density has, in addition to Lorentz invariance, a further invariance under the phase transformation

$$\psi(x) \longrightarrow \psi'(x) = e^{-i\alpha}\psi(x), \quad \bar{\psi}(x) \longrightarrow \bar{\psi}'(x) = e^{i\alpha}\bar{\psi}(x). \tag{10.122}$$

α being a constant parameter. In Sect. 10.2.1, we have referred to analogous transformations on a complex scalar field as *global* $U(1)$ transformations, the term global refers to the property of α of being constant. This is indeed the same invariance exhibited by the Klein–Gordon Lagrangian of a complex scalar field and leads to conservation of a charge according to Noether theorem.

Let us compute the energy-momentum tensor

$$\begin{aligned} T^{\nu\mu} &= \frac{1}{c}\left[\frac{\partial \mathcal{L}}{\partial \partial_\nu \psi(x)} \partial^\mu \psi(x) + \partial^\mu \bar{\psi}(x) \frac{\partial \mathcal{L}}{\partial \partial_\nu \bar{\psi}(x)} - \eta^{\mu\nu}\mathcal{L} \right] \\ &= \frac{1}{c}\left[i\frac{\hbar c}{2} \left(\bar{\psi}\gamma^\nu \partial^\mu \psi - \partial^\mu \bar{\psi}\gamma^\nu \psi \right) - \eta^{\mu\nu}\mathcal{L} \right]. \end{aligned} \tag{10.123}$$

We observe that the Lagrangian density is zero on spinors satisfying the Dirac equation. We may therefore write

$$T^{\nu\mu} = i\frac{\hbar}{2} \left(\bar{\psi}\gamma^\nu \partial^\mu \psi - \partial^\mu \bar{\psi}\gamma^\nu \psi \right). \tag{10.124}$$

This tensor is not symmetric. We can however verify that the divergences of $T^{\mu\nu}$ with respect to both indices vanish:

$$\partial_\mu T^{\nu\mu} = \partial_\mu T^{\mu\nu} = 0, \tag{10.125}$$

The latter equality is a consequence of the Noether theorem, being μ the index of the conserved current. As for the former, it is easily proven as follows:

$$\partial_\mu T^{\nu\mu} = i\frac{\hbar}{2} \left(\partial_\mu \bar{\psi}\gamma^\nu \partial^\mu \psi + \bar{\psi}\gamma^\nu \Box \psi - \Box \bar{\psi}\gamma^\nu \psi - \partial_\mu \bar{\psi}\gamma^\nu \partial^\mu \psi \right) = 0,$$

where we have used the Klein–Gordon equation for ψ and $\bar{\psi}$. Using property (10.125) we can define a symmetric energy momentum-tensor $\Theta^{\mu\nu}$ simply as the symmetric part of $T^{\mu\nu}$:

$$\Theta^{\mu\nu} = \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}), \tag{10.126}$$

since (10.125) guarantee that $\partial_\mu \Theta^{\mu\nu} = 0$. The four-momentum of the spinor field

$$P^\mu = \int_V d^3\mathbf{x} T^{0\mu},$$

has the following form

$$P^\mu = i\frac{\hbar}{2}\int_V d^3\mathbf{x}\left(\bar{\psi}\gamma^0\partial^\mu\psi - \partial^\mu\bar{\psi}\gamma^0\psi\right), \tag{10.127}$$

while the field Hamiltonian $H = cp^0$ reads

$$H = i\frac{\hbar}{2}\int_V d^3\mathbf{x}\left(\psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi\right). \tag{10.128}$$

Using the Dirac equation and integrating by parts, we can easily prove that the right hand side is the sum of two equal terms:

$$\begin{aligned} i\hbar\int_V d^3\mathbf{x}\dot{\bar{\psi}}\gamma^0\psi &= -i\hbar c\int_V d^3\mathbf{x}\partial_i\bar{\psi}\gamma^i\psi - mc^2\int_V d^3\mathbf{x}\bar{\psi}\psi \\ &= i\hbar c\int_V d^3\mathbf{x}\bar{\psi}\gamma^i\partial_i\psi - mc^2\int_V d^3\mathbf{x}\bar{\psi}\psi = -i\hbar\int_V d^3\mathbf{x}\bar{\psi}\gamma^0\dot{\psi}, \end{aligned}$$

so that the Hamiltonian can also be written in the following simpler form:

$$H = i\hbar\int_V d^3\mathbf{x}\psi^\dagger\dot{\psi}. \tag{10.129}$$

Let us now compute the conjugate momenta of the Hamiltonian formalism:

$$\pi(x) = \frac{\partial\mathcal{L}(x)}{\partial\dot{\psi}(x)} = i\frac{\hbar}{2}\psi^\dagger(x), \tag{10.130}$$

$$\pi^\dagger(x) = \frac{\partial\mathcal{L}(x)}{\partial\dot{\psi}^\dagger(x)} = -i\frac{\hbar}{2}\psi(x). \tag{10.131}$$

We note that from these equations it follows that the canonical variables $\pi, \psi, \pi^\dagger, \psi^\dagger$ *are not independent*: $\pi^\dagger \propto \psi$, $\pi \propto \psi^\dagger$. In view of the quantization of the Dirac field, we need to deal with independent canonical variables. It is useful, in this respect, to redefine the Lagrangian density in the following way:

$$\mathcal{L} = i\hbar c\bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) - mc^2\bar{\psi}(x)\psi(x). \tag{10.132}$$

The reader can easily verify that the above expression differs from the previous definition (10.119) by a divergence. We then define, as the only independent variables, the components of $\psi(x)$, so that the corresponding conjugate momenta read

$$\pi(x) = \frac{\partial \mathcal{L}(x)}{\partial \dot{\psi}} = i\hbar\psi^{\dagger}(x). \tag{10.133}$$

From the canonical Poisson brackets (8.225) and (8.226) and the above expression of between $\pi(x)$, we find:

$$\left\{\psi^{\alpha}(\mathbf{x}, t), \psi^{\dagger}_{\beta}(\mathbf{y}, t)\right\} = -\frac{i}{h}\delta^{3}(\mathbf{x} - \mathbf{y})\delta^{\alpha}_{\beta}, \tag{10.134}$$

$$\left\{\psi^{\alpha}(\mathbf{x}, t), \psi^{\beta}(\mathbf{y}, t)\right\} = \left\{\psi^{\dagger}_{\alpha}(\mathbf{x}, t), \psi^{\dagger}_{\beta}(\mathbf{y}, t)\right\} = 0. \tag{10.135}$$

It is convenient to rewrite the Hamiltonian H in (10.129) using Dirac equation (10.53):

$$H = i\hbar \int_V d^3\mathbf{x}\psi^{\dagger}\dot{\psi} = \int_V d^3\mathbf{x}\psi^{\dagger}[-i\hbar c\boldsymbol{\alpha}^i\partial_i + mc^2\boldsymbol{\beta}]\psi. \tag{10.136}$$

The reader can verify that the Hamiltonian density in the above formula can be written in the form:

$$\mathcal{H} = \pi_{\alpha}\psi^{\alpha} - \mathcal{L}. \tag{10.137}$$

We can also verify that the Hamilton equation

$$\dot{\pi}^{\dagger}(x) = -\frac{\delta H}{\delta\psi^{\dagger}(x)} = -\left[-i\hbar c\boldsymbol{\alpha}^i\partial_i + mc^2\boldsymbol{\beta}\right]\psi, \tag{10.138}$$

coincides with the Dirac equation

$$i\hbar\dot{\psi} = (-i\hbar c\boldsymbol{\alpha}^i\partial_i + mc^2\boldsymbol{\beta})\psi.$$

10.6 Plane Wave Solutions to the Dirac Equation

We now examine solutions to the Dirac equation having definite values of energy and momentum. A spinor field with *definite four-momentum* $p = (p^{\mu})$ and spin r, must have the general plane-wave form given in (9.113):

$$\psi_{p,r}(x) = c_p w(p, r)e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x} - Et)} = c_p w(p, r)e^{-\frac{i}{\hbar}p\cdot x}, \tag{10.139}$$

where $w(p, r)$ is a four-component Dirac spinor and c_p a Lorentz invariant normalization factor, to be fixed later. Inserting (10.139) into (10.62), and using the short hand notation $\not{p} \equiv \gamma^{\mu}p_{\mu}$, we find that the generic spinor $w(p)$ satisfies the equation

$$(\not p - mc)\, w(p,r) = 0. \tag{10.140}$$

where $p^\mu = \left(\frac{E}{c}, \mathbf{p}\right)$. If we decompose $w(p,r)$ into two-dimensional spinors as in (10.69) and use the representation (10.64) of the γ-matrices (10.140) becomes:

$$\begin{pmatrix} \frac{E}{c} - mc & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -\frac{E}{c} - mc \end{pmatrix} \begin{pmatrix} \varphi \\ \chi \end{pmatrix} = 0. \tag{10.141}$$

We have shown that each component of $\psi(x)$ is in particular solution to the Klein–Gordon equation (10.59) which implements the mass-shell condition. This can be also verified by multiplying (10.140) to the left by the matrix $(\not p + mc)$:

$$(\not p + mc)(\not p - mc)w(p,r) = (\not p^2 + mc\ \not p - mc\ \not p - m^2c^2)w(p,r) = 0.$$

Using the anti-commutation properties of the γ^μ-matrices we find

$$\not p^2 = \gamma^\mu\gamma^\nu p_\mu p_\nu = \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)p_\mu p_\nu = \eta^{\mu\nu}p_\mu p_\nu = p^2, \tag{10.142}$$

which implies

$$(\not p + mc)(\not p - mc)w(p,r) = (p^2 - m^2c^2)w(p,r) = 0, \tag{10.143}$$

namely the mass-shell condition. As noticed earlier, the Klein–Gordon equation contains negative energy solutions besides the positive energy ones:

$$\frac{E^2}{c^2} = \mathbf{p}^2 + m^2c^2 \Rightarrow E = \pm E_{\mathbf{p}} = \pm\sqrt{|\mathbf{p}|^2c^2 + m^2c^4}. \tag{10.144}$$

The problem of interpreting such solutions, as already mentioned in the case of the complex scalar field, will be resolved by the *field quantization* which associates them with operators creating *antiparticles*. We write the solutions with $E = \pm E_{\mathbf{p}}$ in the following form:

$$\psi^{(+)}_{\mathbf{p},r}(x) \equiv c_p w((E_{\mathbf{p}}/c, \mathbf{p}), r)e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x} - E_{\mathbf{p}}t)} = c_p u(p,r)e^{-\frac{i}{\hbar}p\cdot x},$$
$$\psi^{(-)}_{\mathbf{p},r}(x) \equiv c_p w((-E_{\mathbf{p}}/c, \mathbf{p}), r)e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x} - Et)} = c_p v((E_{\mathbf{p}}/c, -\mathbf{p}), r)e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x} + E_{\mathbf{p}}t)},$$

where we have defined $u(p,r) \equiv w\left(\left(\frac{E_{\mathbf{p}}}{c}, \mathbf{p}\right), r\right)$, $v\left(\left(\frac{E_{\mathbf{p}}}{c}, -\mathbf{p}\right), r\right) \equiv w\left(\left(-\frac{E_{\mathbf{p}}}{c}, \mathbf{p}\right), r\right)$. We shall choose the normalization factor c_p to be: $c_p \equiv \sqrt{\frac{mc^2}{E_{\mathbf{p}}V}}$. Note that the exponent in the definition of $\psi^{(-)}_{\mathbf{p},r}$ acquires a Lorentz-invariant form if we switch $\mathbf{p}$ into $-\mathbf{p}$. We can then write:

$$\psi^{(+)}_{\mathbf{p},r}(x) \equiv \sqrt{\frac{mc^2}{E_{\mathbf{p}}V}}u(p,r)e^{-\frac{i}{\hbar}p\cdot x}, \tag{10.145}$$

$$\psi^{(-)}_{-\mathbf{p},r}(x) \equiv \sqrt{\frac{mc^2}{E_{\mathbf{p}}V}}\, v(p,r) e^{\frac{i}{\hbar} p\cdot x}. \tag{10.146}$$

In the above solutions we have defined $p = (p^\mu) = \left(\frac{E_{\mathbf{p}}}{c}, \mathbf{p}\right)$ so that (10.146) describes a negative-energy state with momentum $-p$, $v(p,r) \equiv w(-p,r)$.

The general solution to the Dirac equation will be expanded in both kinds of solutions, and have the following form:

$$\psi(x) = \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} V \sum_{r=1}^{2} \left(c(\mathbf{p},r)\psi^{(+)}_{\mathbf{p},r}(x) + d(-\mathbf{p},r)^* \psi^{(-)}_{\mathbf{p},r}(x) \right),$$

where c, d are complex numbers representing the components of $\psi(x)$ relative to the complete set of solutions $\psi^{(\pm)}_{\mathbf{p},r}(x)$. By changing $\mathbf{p}$ into $-\mathbf{p}$ in the integral of the second term on the right hand side, we have:

$$\begin{aligned}\psi(x) &= \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} V \sum_{r=1}^{2} \left(c(\mathbf{p},r)\psi^{(+)}_{\mathbf{p},r}(x) + d(\mathbf{p},r)^* \psi^{(-)}_{-\mathbf{p},r}(x) \right) \\ &= \int \frac{d^3\mathbf{p}}{(2\pi\hbar)^3} \sqrt{\frac{mc^2 V}{E_{\mathbf{p}}}} \sum_{r=1}^{2} \left(c(\mathbf{p},r) u(p,r) e^{-\frac{i}{\hbar} p\cdot x} + d(\mathbf{p},r)^* v(p,r) e^{\frac{i}{\hbar} p\cdot x} \right).\end{aligned} \tag{10.147}$$

We need now to explicitly construct the spinors $u(p,r)$, $v(p,r)$. Being $u(p,r) = w(p,r)$ and $v(p,r) = w(-p,r)$, the equation for $u(p,r)$ is the same as (10.140), while the one for $v(p,r)$ is obtained from (10.140) by replacing $p \to -p$:

$$(\not{p} - mc)u(p,r) = 0; \quad (\not{p} + mc)v(p,r) = 0. \tag{10.148}$$

The Lorentz covariance of the above equations implies that $S(\mathbf{\Lambda})u(p,r)$ and $S(\mathbf{\Lambda})v(p,r)$ must be a combination of $u(\mathbf{\Lambda}p, s)$ and $v(\mathbf{\Lambda}p, s)$,[8] with coefficients given by the rotation $\mathcal{R}(\mathbf{\Lambda}, p)^s{}_r$ of (10.107), or (10.110) for massless particles, according to our discussion in Sect. 9.4.1:

$$\begin{aligned} S(\mathbf{\Lambda})u(p,r) &= \mathcal{R}(\mathbf{\Lambda}, p)_r^{r'} u(\mathbf{\Lambda}p, r') \\ S(\mathbf{\Lambda})v(p,r) &= \mathcal{R}(\mathbf{\Lambda}, p)_r^{r'} v(\mathbf{\Lambda}p, r'). \end{aligned} \tag{10.149}$$

These are nothing but the transformation properties derived in (9.118). In the frame $\mathcal{S}_0$ in which the momentum p is the standard one $\bar{p}$, $u(\bar{p}, r)$ and $v(\bar{p}, r)$ transform

[8] This can be easily ascertained by multiplying both (10.148) to the left by $S(\mathbf{\Lambda})$. We find that $S(\mathbf{\Lambda})u(p,r)$ and $S(\mathbf{\Lambda})v(p,r)$ satisfy the following equations: $(S(\mathbf{\Lambda})\not{p}S(\mathbf{\Lambda})^{-1} - mc)S(\mathbf{\Lambda})u(p,r) = 0$ and $(S(\mathbf{\Lambda})\not{p}S(\mathbf{\Lambda})^{-1} + mc)S(\mathbf{\Lambda})v(p,r) = 0$. Next we use property (10.89) and invariance of the Lorentzian scalar product $\gamma \cdot p \equiv \gamma^\mu p_\mu = \not{p}$ to write $S(\mathbf{\Lambda})\not{p}S(\mathbf{\Lambda})^{-1} = \not{p}' = \gamma^\mu p'_\nu$, where $p' = \mathbf{\Lambda}p$. Thus the transformed spinors satisfy (10.148) with the transformed momentum p', and consequently, should be a combination of $u(p', s)$ and $v(p', s)$, respectively.

covariantly under the action of the spin group. Let us construct them in this frame and then extend their definition to a generic one.

Consider a massive particle, $m \neq 0$, and let us first examine the solutions of the coupled system (10.141) in the rest frame $\mathcal{S}_0$, where $\mathbf{p} = \mathbf{0}$, namely $\bar{p} = (mc, \mathbf{0})$. Equation (10.141) becomes:

$$\left(E - mc^2\right)\varphi = 0; \quad \left(E + mc^2\right)\chi = 0. \tag{10.150}$$

Then we have either

$$E = E_{\mathbf{p}=0} = mc^2; \quad \varphi \neq 0, \chi = 0,$$

or

$$E = -E_{\mathbf{p}=0} = -mc^2; \quad \varphi = 0, \chi \neq 0.$$

The non zero spinors in the two cases can be chosen arbitrarily. We choose them to be eigenvectors of σ^3:

$$\varphi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \varphi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{10.151}$$

In $\mathcal{S}_0$ we can then write the positive and negative energy solutions in the momentum representation as

$$u(\mathbf{0}, r) \equiv u(\bar{p}, r) = \begin{pmatrix} \varphi_r \\ \mathbf{0} \end{pmatrix}; \quad v(\mathbf{0}, r) \equiv v(\bar{p}, r) = \begin{pmatrix} \mathbf{0} \\ \varphi_r \end{pmatrix} \quad r = 1, 2, \tag{10.152}$$

where $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since the φ_r are eigenstates of σ^3, the *rest frame* solutions $u(\mathbf{0}, r)$ and $v(\mathbf{0}, r)$ are eigenstates of the operator:

$$\Sigma^3 = \begin{pmatrix} \frac{\hbar}{2}\sigma^3 & 0 \\ 0 & \frac{\hbar}{2}\sigma^3 \end{pmatrix}, \tag{10.153}$$

corresponding to the eigenvalues $\pm\hbar/2$.

Once the solutions in the rest frame are given we may construct the solutions $u(p, r)$ and $v(p, r)$ of the Dirac equation in a generic frame $\mathcal{S}$ where $\mathbf{p} \neq 0$ as follows:

$$u(p, r) = \frac{\not{p} + mc}{\sqrt{2m(mc^2 + E_{\mathbf{p}})}} u(\mathbf{0}, r), \tag{10.154}$$

$$v(p, r) = \frac{-\not{p} + mc}{\sqrt{2m(mc^2 + E_{\mathbf{p}})}} v(\mathbf{0}, r). \tag{10.155}$$

The denominators appearing in (10.154) and (10.155) are normalization factors determined in such a way that the spinors $u(p, r), v(p, r)$ obey simple normalization conditions (see (10.168)–(10.169) of the next section). It is straightforward to show that $u(p,r)$ and $v(p,r)$ satisfy (10.148) by using the properties

$$(\not p + mc)(\not p - mc) = (\not p - mc)(\not p + mc) = p^2 - m^2c^2 + mc\ \not p - mc\ \not p = p^2 - m^2c^2 = 0, \tag{10.156}$$

which descend from (10.142).

Using the representation (10.64) of the γ-matrices and the explicit form of $\not p$, we obtain $u(p,r)$ and $v(p,r)$ in components:

$$u(p,r) = \begin{pmatrix} \sqrt{\frac{E_{\mathbf{p}}+mc^2}{2mc^2}}\varphi_r \\ \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{\sqrt{2m(E_{\mathbf{p}}+mc^2)}}\varphi_r \end{pmatrix}; \quad v(p,r) = \begin{pmatrix} \frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{\sqrt{2m(E_{\mathbf{p}}+mc^2)}\varphi_r} \\ \sqrt{\frac{E_{\mathbf{p}}+mc^2}{2mc^2}}\varphi_r \end{pmatrix}. \tag{10.157}$$

Let us show that the above vectors transform as in (10.149) with respect to rotations $\boldsymbol{\Lambda}_R$:

$$S(\boldsymbol{\Lambda}_R)u(p,r) = e^{\frac{i}{\hbar}\theta^i\Sigma_i}u(p,r) = \begin{pmatrix} \sqrt{\frac{E_{\mathbf{p}}+mc^2}{2mc^2}}\mathbf{S}(\theta^i)\varphi_r \\ \frac{\mathbf{S}(\theta^i)\mathbf{p}\cdot\boldsymbol{\sigma}}{\sqrt{2m(E_{\mathbf{p}}+mc^2)}}\varphi_r \end{pmatrix}$$
$$= \begin{pmatrix} \sqrt{\frac{E_{\mathbf{p}}+mc^2}{2mc^2}}\varphi'_r \\ \frac{\mathbf{S}(\theta^i)\mathbf{p}\cdot\boldsymbol{\sigma}\mathbf{S}(\theta^i)^{-1}}{\sqrt{2m(E_{\mathbf{p}}+mc^2)}}\varphi'_r \end{pmatrix}, \tag{10.158}$$

where:

$$\varphi'_r \equiv \mathbf{S}(\theta^i)\varphi_r = S(\theta^i)^s{}_r\varphi_s = \mathcal{R}^s{}_r\varphi_s. \tag{10.159}$$

Let us now use the property of the Pauli matrices to transform under conjugation by an SU(2) matrix $\mathbf{S}(\boldsymbol{\theta})$, $\boldsymbol{\theta} \equiv (\theta^i)$, as the components of a three-dimensional vector $\boldsymbol{\sigma} \equiv (\sigma_i)$ under a corresponding rotation $\mathbf{R}(\boldsymbol{\theta})$, see Appendix (F):

$$\mathbf{S}(\boldsymbol{\theta})^{-1}\sigma_i\mathbf{S}(\boldsymbol{\theta}) = R(\boldsymbol{\theta})_i{}^j\sigma_j \Rightarrow \mathbf{S}(\boldsymbol{\theta})\sigma_i\mathbf{S}(\boldsymbol{\theta})^{-1} = R(\boldsymbol{\theta})^{-1}{}_i{}^j\sigma_j. \tag{10.160}$$

We can then write:

$$\mathbf{S}(\boldsymbol{\theta})\mathbf{p}\cdot\boldsymbol{\sigma}\mathbf{S}(\boldsymbol{\theta})^{-1} = \mathbf{p}\cdot\left(\mathbf{R}(\boldsymbol{\theta})^{-1}\boldsymbol{\sigma}\right) = \mathbf{p}'\cdot\boldsymbol{\sigma}, \tag{10.161}$$

where $\mathbf{p}' \equiv \mathbf{R}(\boldsymbol{\theta})\mathbf{p}$. Since $\boldsymbol{\Lambda}_R p = (p^0, \mathbf{p}')$, we conclude that

$$S(\boldsymbol{\Lambda}_R)u(p,r) = \mathcal{R}^s{}_r \begin{pmatrix} \sqrt{\frac{E_{\mathbf{p}}+mc^2}{2mc^2}}\varphi_s \\ \frac{\mathbf{p}'\cdot\boldsymbol{\sigma}}{\sqrt{2m(E_{\mathbf{p}}+mc^2)}}\varphi_s \end{pmatrix} = \mathcal{R}^s{}_r u(\boldsymbol{\Lambda}_R p, s). \tag{10.162}$$

A similar derivation can be done for $v(p, r)$. If $\boldsymbol{\Lambda}$ is a boost, of the form $\boldsymbol{\Lambda} = \exp\left(\frac{i}{\hbar}\omega_{0i} J^{0i}\right)$, the corresponding representation on the spinors reads $S(\boldsymbol{\Lambda}) = \exp\left(\frac{i}{\hbar}\omega_{0i}\Sigma^{0i}\right)$. The resulting SU(2) rotation $\mathcal{R}(\boldsymbol{\Lambda}, p)$, which we are not going to derive, is the Wigner rotation.

We note that $u(p, r)$ and $v(p, r)$ are not eigenstates of the third component of the spin operator Σ^3 (10.153) except in the special case of $p^1 = p^2 = 0$, $p^3 \neq 0$. This can be explained in light of the discussion done in Sect. 9.4.1 about little groups. The solutions $u(p, r)$ and $v(p, r)$, for a fixed p, transform as doublets with respect to the little group of the momentum p, which we have denoted by $G_p^{(0)}$: The action of $G_p^{(0)}$ on the solutions $u(p, r)$ and $v(p, r)$, according to (10.149), does not affect their dependence on p, and only amounts to an SU(2)-transformation on the index r. This group is related to the little group $G^{(0)} = \mathrm{SU}(2)$ of $\bar{p} = (mc, \mathbf{0})$, generated by the Σ^i matrices as follows: $G_p^{(0)} = \boldsymbol{\Lambda}_p \cdot \mathrm{SU}(2) \cdot \boldsymbol{\Lambda}_p^{-1}$. This means that its generators are $\Sigma_i' = S(\boldsymbol{\Lambda}_p)\Sigma_i S(\boldsymbol{\Lambda}_p)^{-1}$. If instead we act on $u(p, r)$ and $v(p, r)$ by means of a $G^{(0)} = \mathrm{SU}(2)$-transformation, it will affect dependence of these fields on p, mapping it into $p' = (p^0, \mathbf{Rp})$. Therefore, if $u(\bar{p}, r)$ and $v(\bar{p}, r)$ are eigenvectors of Σ_3, $u(p, r)$ and $v(p, r)$ will be eigenvectors of Σ_3'.

In Sect. 9.4.1 of last chapter, a general method was applied to the construction of the single-particle quantum states $|p, r\rangle$ acted on by a unitary irreducible representation of the Lorentz group. The method consisted in first constructing the states of the particle $|\bar{p}, r\rangle$ in some special frame S_0 in which the momentum of the particle is the standard one $\bar{p}$, and on which an irreducible representation $\mathcal{R}$ of the little group $G^{(0)}$ of $\bar{p}$ acts ($\bar{p} = (mc, \mathbf{0})$ and $G^{(0)} = \mathrm{SU}(2)$ for massive particles, while $\bar{p} = (E, E, 0, 0)/c$ and $G^{(0)}$ is effectively SO(2) for massless particles). A generic state $|p, r\rangle$ is then constructed by acting on $|\bar{p}, r\rangle$ by means of $U(\boldsymbol{\Lambda}_p)$, see (9.111), that is the representative on the quantum states of the simple Lorentz boost $\boldsymbol{\Lambda}_p$ connecting $\bar{p}$ to p: $p = \boldsymbol{\Lambda}_p\bar{p}$. This suffices to define the representative $U(\boldsymbol{\Lambda})$ of a generic Lorentz transformation, see (9.112). In this section we have applied this prescription to the construction of both the positive and negative energy eigenstates of the momentum operators. The role of $|p, r\rangle$ is now played by the spinors $u(p, r)$, $v(p, r)$, and that of $U(\boldsymbol{\Lambda})$ by the matrix $S(\boldsymbol{\Lambda})$, as it follows by comparing (10.149) with (9.112). It is instructive at this point to show that the expressions for $u(p, r)$, $v(p, r)$ given in (10.154) or, equivalently, (10.157), for massive fermions, could have been obtained from the corresponding spinors $u(\mathbf{0}, r)$, $v(\mathbf{0}, r)$ in S_0 using the prescription (9.111), namely by acting on them through the Lorentz boost $S(\boldsymbol{\Lambda}_p)$:

$$u(p, r) = S(\boldsymbol{\Lambda}_p)u(\mathbf{0}, r); \quad v(p, r) = S(\boldsymbol{\Lambda}_p)v(\mathbf{0}, r). \tag{10.163}$$

This is readily proven using the matrix form (10.118) of $S(\boldsymbol{\Lambda}_p)$ derived in Sect. 10.4.4 and the definition of $u(\mathbf{0}, r)$, $v(\mathbf{0}, r)$ in (10.152). The matrix product on the right hand side of (10.163) should then be compared with the matrix form of $u(p, r)$, $v(p, r)$ in (10.157).

10.6.1 Useful Properties of the $u(p, r)$ and $v(p, r)$ Spinors

In the following we shall prove some properties of the spinors $u(p, r)$ and $v(p, r)$ describing solutions with definite four-momentum.

• Let us compute the Dirac conjugates of $u(p, r)$ e $v(p, r)$:

$$\begin{aligned}\bar{u}(p, r) &= u^\dagger(p, r)\gamma^0 = u^\dagger(\mathbf{0}, r)\frac{\not{p}^\dagger + mc}{\sqrt{2m(E_\mathbf{p} + mc^2)}}\gamma^0\\ &= u^\dagger(\mathbf{0}, r)\gamma^0\gamma^0\frac{\not{p}^\dagger + mc}{\sqrt{2m(E_\mathbf{p} + mc^2)}}\gamma^0\\ &= \bar{u}(\mathbf{0}, r)\frac{\not{p} + mc}{\sqrt{2m(E_\mathbf{p} + mc^2)}}.\end{aligned} \tag{10.164}$$

In an analogous way one finds:

$$\bar{v}(p, r) = \bar{v}(\mathbf{0}, r)\frac{-\not{p} + mc}{\sqrt{2m(E_\mathbf{p} + mc^2)}}. \tag{10.165}$$

Recalling the property (10.156), from (10.164) and (10.165) we obtain the equations of motion obeyed by the Dirac spinors $\bar{u}(p, r)$ e $\bar{v}(p, r)$:

$$\begin{aligned}\bar{u}(p, r)(\not{p} - mc) &= 0,\\ \bar{v}(p, r)(\not{p} + mc) &= 0.\end{aligned} \tag{10.166}$$

• Next we use the relations:

$$\begin{aligned}(\not{p} + mc)^2 &= 2mc(\not{p} + mc),\\ (\not{p} - mc)^2 &= 2mc(-\not{p} + mc),\end{aligned} \tag{10.167}$$

which follow from (10.142) and the mass-shell condition $p^2 = m^2c^2$, to compute $\bar{u}(p, r)u(p, r')$:

$$\begin{aligned}\bar{u}(p, r)u(p, r') &= \frac{2mc}{2m(E_\mathbf{p} + mc^2)}\bar{u}(\mathbf{0}, r)(\not{p} + mc)u(\mathbf{0}, r')\\ &= \frac{c}{E_\mathbf{p} + mc^2}(\varphi_r, 0, 0)(\not{p} + mc)\begin{pmatrix}0\\0\\\varphi_{r'}\end{pmatrix}\\ &= \varphi_r \cdot \varphi_{r'} = \delta_{rr'},\end{aligned} \tag{10.168}$$

With analogous computations one also finds:

$$\begin{aligned}\bar{v}(p,r)v(p,r') &= \frac{c}{E_{\mathbf{p}}+mc^2}\bar{v}(\mathbf{0},r)(-\not{p}+mc)v(\mathbf{0},r')\\ &= \frac{c}{E_{\mathbf{p}}+mc^2}(0,0,-\varphi_r)(-\not{p}+mc)\begin{pmatrix}0\\0\\\varphi_{r'}\end{pmatrix}\\ &= -\delta_{rr'},\end{aligned} \tag{10.169}$$

and moreover

$$\begin{aligned}\bar{u}(p,r)v(p,r') &\propto \; \bar{u}(\mathbf{0},r)(\not{p}+mc)(-\not{p}+mc)v(\mathbf{0},r')\\ &= 0 = \bar{v}(p,r)u(p,r').\end{aligned} \tag{10.170}$$

Summarizing, we have obtained the relations

$$\begin{aligned}&\bar{u}(p,r)u(p,r') = \delta_{rr'}, = -\bar{v}(p,r)v(p,r'),\\ &\bar{u}(p,r)v(p,r') = 0.\end{aligned} \tag{10.171}$$

• Next we show that:

$$u^\dagger(p,r)u(p,r') = \frac{E_{\mathbf{p}}}{mc^2}\delta_{rr'} \geq 0, \tag{10.172}$$

$$v^\dagger(p,r)v(p,r') = \frac{E_{\mathbf{p}}}{mc^2}\delta_{rr'} \geq 0. \tag{10.173}$$

Indeed, using the Dirac equation $\not{p}u = mcu$, and $\bar{u}\not{p} = mc\bar{u}$, we find

$$\begin{aligned}u^\dagger(p,r)u(p,r') = \bar{u}(p,r)\gamma^0 u(p,r') &= \bar{u}(p,r)\frac{m\gamma^0+m\gamma^0}{2m}u(p,r')\\ &= \bar{u}(p,r)\frac{\not{p}\gamma^0+\gamma^0\not{p}}{2mc}u(p,r').\end{aligned}$$

Using now the property

$$\not{p}\gamma^0+\gamma^0\not{p} = \{\not{p},\gamma^0\} = p_\mu\{\gamma^\mu,\gamma^0\} = 2\eta^{\mu 0}p_\mu = \frac{2E_{\mathbf{p}}}{c},$$

the last term, can be rewritten as follows:

$$\frac{E_{\mathbf{p}}}{mc^2}\bar{u}(p,r)u(p,r') = \frac{E_{\mathbf{p}}}{mc^2}\delta_{rr'}.$$

so that (10.172) is retrieved. Equation (10.173) is obtained in an analogous way.

We conclude that $u^\dagger(p,r)u(p,r')$ and $v^\dagger(p,r)v(p,r')$ *are not Lorentz invariant quantities*, since they transform as $E_{\mathbf{p}}$, that is as the time component of a four-vector.

This agrees with the previous result that $J^\mu = \bar{\psi}\gamma^\mu\psi$ is a four-vector whose time component is $J^0 = \psi^\dagger\psi > 0$.

We can also prove the following orthogonality condition:

$$u(\mathbf{p}, r)^\dagger v(-\mathbf{p}, s) = 0, \tag{10.174}$$

where we have used the short-hand notation $u(\mathbf{p}, r) \equiv u((E_{\mathbf{p}}/c, \mathbf{p}), r)$, $v(\mathbf{p}, r) \equiv v((E_{\mathbf{p}}/c, \mathbf{p}), r)$. To prove the above equation we use the Dirac equation for $v(-\mathbf{p}, s)$: $\not{p}' v(-\mathbf{p}, s) = -mcv(-\mathbf{p}, s)$, where $p' \equiv (E_{\mathbf{p}}/c, -\mathbf{p})$. We can then write:

$$\begin{aligned} u(\mathbf{p}, r)^\dagger v(-\mathbf{p}, s) &= \bar{u}(\mathbf{p}, r)\gamma^0 v(-\mathbf{p}, s) = \frac{1}{2mc}\bar{u}(\mathbf{p}, r)(\not{p}\gamma^0 - \gamma^0\not{p}')v(-\mathbf{p}, s) \\ &= \frac{1}{2mc}\bar{u}(\mathbf{p}, r)(p_i\gamma^i\gamma^0 + \gamma^0\gamma^i p_i)v(-\mathbf{p}, s) = 0. \end{aligned} \tag{10.175}$$

From property (10.174) it also follows that *positive and negative energy states are represented by mutually orthogonal spinors if they have the same momentum:*

$$\left[\psi_{\mathbf{p}}^{(+)}(x)\right]^\dagger \psi_{\mathbf{p}}^{(-)}(x) = 0. \tag{10.176}$$

Recalling from (10.145) and (10.146) that

$$\psi_{\mathbf{p},r}^{(+)}(x) = c_p u(\mathbf{p}, r)e^{-\frac{i}{\hbar}(E_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x})}; \quad \psi_{\mathbf{p},r}^{(-)}(x) = c_p v(-\mathbf{p}, r)e^{\frac{i}{\hbar}(E_{\mathbf{p}}t + \mathbf{p}\cdot\mathbf{x})},$$

from the orthogonality condition (10.174) it indeed follows that

$$\psi_{\mathbf{p},r}^{(+)}(x)^\dagger \psi_{\mathbf{p},s}^{(-)}(x) = |c_p|^2 u^\dagger(\mathbf{p}, r)v(-\mathbf{p}, s)e^{\frac{2i}{\hbar}E_{\mathbf{p}}t} = 0. \tag{10.177}$$

Having fixed the normalization factor c_p in (10.139) to be $\sqrt{\frac{mc^2}{VE_{\mathbf{p}}}}$, we now observe that (10.172) and (10.173) represent the right normalization (9.116) of the u and v vectors in order for the corresponding positive and negative energy solutions $\psi_{\mathbf{p},r}^{(\pm)}(x)$ to be normalized as in (9.54):

$$\left(\psi_{\mathbf{p},r}^{(\pm)}, \psi_{\mathbf{p}',r'}^{(\pm)}\right) = \int d^3\mathbf{x}\psi_{\mathbf{p},r}^{(\pm)}(x)^\dagger\psi_{\mathbf{p}',r'}^{(\pm)}(x) = \frac{(2\pi\hbar)^3}{V}\delta^3(\mathbf{p} - \mathbf{p}')\delta_{rr'},$$

as the reader can easily verify. Similarly, using the orthogonality condition (10.174), which applies to the above expression only when $\mathbf{p}' = \mathbf{p}$, we can show that positive and negative energy solutions are mutually orthogonal:

$$\begin{aligned} \left(\psi_{\mathbf{p},r}^{(+)}, \psi_{\mathbf{p}',r'}^{(-)}\right) &= \int d^3\mathbf{x}|c_p|^2 u(\mathbf{p}, r)^\dagger v(-\mathbf{p}', r')e^{\frac{i}{\hbar}(E_{\mathbf{p}} + E'_{\mathbf{p}})t}e^{-\frac{i}{\hbar}(\mathbf{p} - \mathbf{p}')\cdot\mathbf{x}} \\ &\propto (2\pi\hbar)^3\delta^3(\mathbf{p} - \mathbf{p}')e^{\frac{2i}{\hbar}E_{\mathbf{p}}t}u(\mathbf{p}, r)^\dagger v(-\mathbf{p}, r') = 0. \end{aligned}$$

- Finally we define *projection operators* $\Lambda_+(\pm p)$ on the positive and negative energy solutions:

$$\Lambda_+(p)^\alpha{}_\beta \equiv \sum_{r=1}^{2} u(p,r)^\alpha \bar{u}(p,r)_\beta, \tag{10.178}$$

$$\Lambda_-(p)^\alpha{}_\beta \equiv -\sum_{r=1}^{2} v(p,r)^\alpha \bar{v}(p,r)_\beta. \tag{10.179}$$

Using the formulae (10.171) we see $\Lambda_\pm(p)$ are indeed projection operators:

$$\Lambda_+(p)u(p,r) = u(p,r); \quad \Lambda_+(p)v(p,r) = 0, \tag{10.180}$$

$$\Lambda_-(p)u(p,r) = 0; \quad \Lambda_-(p)v(p,r) = v(p,r). \tag{10.181}$$

The explicit form of $\Lambda_\pm$ is immediately derived from (10.167) since they express the fact that $\not{p} \pm mc$ are proportional to projection operators. Thus we have:

$$\Lambda_+(p) = \frac{1}{2mc}(\not{p} + mc), \tag{10.182}$$

$$\Lambda_-(p) = -\frac{1}{2mc}(\not{p} - mc). \tag{10.183}$$

10.6.2 Charge Conjugation

We show the existence of an operator in the Dirac relativistic theory which transforms *positive energy* solutions into *negative energy* solutions, and viceversa. One can prove on general grounds that that there exists a matrix in spinor space, called the *charge-conjugation matrix* with the following properties

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T; \quad C^T = -C; \quad C^\dagger = C^{-1}. \tag{10.184}$$

In the standard representation we may identify the C matrix as

$$C = i\gamma^2\gamma^0 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}. \tag{10.185}$$

Given a Dirac field $\psi(x)$, we define its *charge conjugate* spinor $\psi^c(x)$ as follows:

$$\psi^c(x) \equiv C\bar{\psi}^T(x). \tag{10.186}$$

The operation which maps $\psi(x)$ into its charge conjugate $\psi^c(x)$ is called *charge-conjugation*. Let us show that charge conjugation is a correspondence between positive and negative energy solutions.

To this end let us consider the positive energy plane wave described by the spinor $u(p,r)$. Its Dirac conjugate $\bar{u}$ will satisfy the following equation:

$$\bar{u}(p)\,(\not{p} - mc) = 0.$$

By transposition we have

$$\left(\gamma_\mu^T p^\mu - mc\right)\bar{u}^T(p) = 0$$

If we now multiply the above equation to the left by the C matrix and use the property (10.184) we obtain

$$(\not{p} + mc)\, C\bar{u}^T(p) = 0, \tag{10.187}$$

which shows that charge-conjugate spinor $u^c(p) = C\bar{u}^T(p)$ satisfies the second of (10.148) and should therefore coincide with a spinor $v(p)$ defining the negative energy solution $\psi^{(-)}_{-\mathbf{p}}$ with opposite momentum $-\mathbf{p}$. Besides changing the value of the momentum, charge-conjugation also reverses the spin orientation. Going, for the sake of simplicity, to the rest frame, where a positive energy solution with spin projection $\hbar/2$ along a given direction, is described by

$$u(\mathbf{0},1) = (1,0,0,0)^T,$$

see (10.152), we find for the charge conjugate spinor $u^c \equiv C\gamma^0 u^*$ (note that $\gamma^{0T} = \gamma^0$)

$$u^c(\mathbf{0},r) = C\gamma^0 u^*(\mathbf{0}, r=1) = (0,0,0,1)^T = v(\mathbf{0}, r=2),$$

that is a negative energy spinor with spin projection $-\hbar/2$. In general the reader can verify that

$$u^c(\mathbf{0},r) = \epsilon_{rs} v(\mathbf{0},s), \tag{10.188}$$

where summation over $s = 1,2$ is understood, and (ϵ_{rs}) is the matrix $i\sigma_2$: $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = -\epsilon_{21} = 1$.

Let us now evaluate $u^c(p,r)$ using the explicit form of $u(p,r)$ given in (10.154):

$$\begin{aligned}
u^c(p,r) &= C\gamma^0 u(p,r)^* = C\gamma^0 \frac{\not{p}^* + mc}{\sqrt{2m(mc^2 + E_{\mathbf{p}})}} u(\mathbf{0},r)^* \\
&= C\frac{\not{p}^T + mc}{\sqrt{2m(mc^2 + E_{\mathbf{p}})}}\gamma^0 u(\mathbf{0},r)^* = \frac{-\not{p} + mc}{\sqrt{2m(mc^2 + E_{\mathbf{p}})}} u^c(\mathbf{0},r) \\
&= \epsilon_{rs}\frac{-\not{p} + mc}{\sqrt{2m(mc^2 + E_{\mathbf{p}})}} v(\mathbf{0},s) = \epsilon_{rs} v(p,s).
\end{aligned} \tag{10.189}$$

In the above derivation we have used the properties $C\ \not{p}^T C^{-1} = -\ \not{p}$ and $\gamma^0 \not{p}^* = \not{p}^T \gamma^0$.

We shall see in the next chapter that, upon quantizing the Dirac field, negative energy solutions $\psi^{(-)}_{-\mathbf{p},r}$ with momentum $-\mathbf{p}$ and a certain spin component (up or down relative to a given direction) are reinterpreted as creation operators of *antiparticles* with positive energy, momentum $\mathbf{p}$ and opposite spin component. Thus the charge conjugation operation can be viewed as the operation which interchanges particles with antiparticles with the same momentum and spin. As far as the electric charge is concerned we need to describe the coupling of a *charge conjugate spinor* to an external electromagnetic field as it was done for the scalar field. This will be discussed in Sect. 10.7. We anticipate that the electric charge of a charge conjugate spinor describing an antiparticle is *opposite* to that of the corresponding particle.

10.6.3 Spin Projectors

In Sect. 10.6.1 we have labeled the spin states of the massive solutions to the Dirac equation by the eigenvalues, in the rest frame, of Σ_3: $u(\mathbf{0}, r)$, $v(\mathbf{0}, r)$, for $r = 1, 2$ correspond to the eigenvalues $+\hbar/2$ and $-\hbar/2$ of Σ_3. This amounts to choosing the two-component vectors φ_r to correspond to the eigenvalues $+1$ and -1 of σ_3. We could have chosen $u(\mathbf{0}, r)$, $v(\mathbf{0}, r)$ to be eigenvectors of the spin-component $\boldsymbol{\Sigma} \cdot \mathbf{n}$ along a generic direction $\mathbf{n}$ in space, $|\mathbf{n}| = 1$. The corresponding eigenvalues would still be $\pm\hbar/2$. Clearly, for generic $\mathbf{n}$, $\boldsymbol{\Sigma} \cdot \mathbf{n}$ is not conserved, namely it does not commute with the Hamiltonian, as proven in Sect. 10.4.4. This is not the case if $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$, in which case the corresponding component of the spin vector defines the helicity $\Gamma = \boldsymbol{\Sigma} \cdot \mathbf{p}/|\mathbf{p}|$ which is indeed conserved.

We now ask whether it is possible to give a *covariant* meaning to the value of the spin orientation along a direction $\mathbf{n}$. We wish in other words to define a Lorentz-invariant operator O_n which reduces to $\boldsymbol{\Sigma} \cdot \mathbf{n}$ in the rest frame, namely such that, if in $\mathcal{S}_0$:

$$(\boldsymbol{\Sigma} \cdot \mathbf{n})u(\mathbf{0}, r) = \varepsilon_r \frac{\hbar}{2} u(\mathbf{0}, r); \quad (\boldsymbol{\Sigma} \cdot \mathbf{n})v(\mathbf{0}, r) = \varepsilon_r \frac{\hbar}{2} v(\mathbf{0}, r), \tag{10.190}$$

where $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, in a generic frame $\mathcal{S}$:

$$O_n u(p, r) = \varepsilon_r \frac{\hbar}{2} u(p, r); \quad O_n v(p, r) = \varepsilon_r \frac{\hbar}{2} v(p, r). \tag{10.191}$$

Clearly, using (10.163), we must have:

$$O_n = S(\boldsymbol{\Lambda}_p)(\boldsymbol{\Sigma} \cdot \mathbf{n})S(\boldsymbol{\Lambda}_p)^{-1} = \boldsymbol{\Sigma}' \cdot \mathbf{n}, \tag{10.192}$$

where Σ'_i are the generators of the little group $G^{(0)}_p \equiv \mathrm{SU}(2)_p$ of p.

We shall however compute O_n in a simpler way, using the *Pauli-Lubanski* four-vector $\hat{W}_\mu$ introduced in Sect. 9.4.2, which, on spinor solutions with definite momentum p^μ, acts by means of the following matrices:

$$W_\mu \equiv -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\Sigma^{\nu\rho}p^\sigma, \tag{10.193}$$

It is useful to write it in a simpler way by introducing the matrix γ^5 (see Appendix G):

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \begin{pmatrix} \mathbf{0} & \mathbf{1}_2 \\ \mathbf{1}_2 & \mathbf{0} \end{pmatrix}. \tag{10.194}$$

Note that γ^5 anticommutes with all the γ^μ-matrices and thus commutes with the Lorentz generators $\Sigma^{\mu\nu}$ which contain products of two γ^μ-matrices. From this we conclude that γ^5 commutes with a generic Lorentz transformation $S(\mathbf{\Lambda})$, since it commutes with its infinitesimal generator.

Using the γ^5 matrix the Pauli-Lubanski four-vector (10.193) takes the simpler form:

$$\begin{aligned} W_\mu &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\left(-\frac{\hbar}{2}\sigma^{\nu\rho}\right)p^\sigma = \frac{\hbar}{4}\epsilon_{\mu\nu\rho\sigma}\sigma^{\nu\rho}p^\sigma \\ &= \frac{i\hbar}{2}\gamma^5\sigma_{\mu\nu}p^\nu = -i\gamma^5\Sigma_{\mu\nu}p^\nu, \end{aligned} \tag{10.195}$$

where we have used the identity

$$\gamma^5\sigma_{\mu\sigma} = -\frac{i}{2}\epsilon_{\mu\sigma\nu\rho}\sigma^{\nu\rho},$$

given in Appendix G, which can be verified by direct computation, starting from the definition of γ^5. Using the Lorentz transformation properties (10.89) of the γ^μ-matrices, and the invariance of the $\epsilon_{\mu\nu\rho\sigma}$-tensor under proper transformations, we can easily verify that W^μ transforms like the γ^μ-matrices:

$$S(\mathbf{\Lambda})W^\mu S(\mathbf{\Lambda})^{-1} = \Lambda^{-1\mu}{}_\nu W^\nu. \tag{10.196}$$

Let us now introduce the four-vector $n^\mu(\mathbf{p}) = (n^0(\mathbf{p}), \mathbf{n}(\mathbf{p}))$ having the following properties:

$$\begin{cases} n^2 = n^\mu n_\mu = -1, \\ n_\mu p^\mu = 0. \end{cases} \tag{10.197}$$

In the rest frame, $\mathbf{p} = 0$ and $E = mc^2 \neq 0$, the previous relations yield:

$$\begin{aligned} n_\mu p^\mu = n^0 E = 0 \Rightarrow n^0 = 0, \\ n^2 = (n^0)^2 - |\mathbf{n}|^2 = -1 \Rightarrow |\mathbf{n}| = 1, \end{aligned} \tag{10.198}$$

that is $n_\mu(\mathbf{p}=\mathbf{0}) = (0, \mathbf{n})$. We may now compute the scalar quantity $n^\mu W_\mu$:

$$\begin{aligned} n^\mu W_\mu &= \frac{i\hbar}{2}\gamma^5 \sigma_{\mu\nu} n^\mu p^\nu - \frac{\hbar}{4}\gamma^5(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) n^\mu p^\nu \\ &= -\frac{\hbar}{4}\gamma^5(2\gamma_\mu\gamma_\nu - 2\eta_{\mu\nu}) n^\mu p^\nu = -\frac{\hbar}{2}\gamma^5\gamma_\mu n^\mu \not{p}. \end{aligned} \tag{10.199}$$

where the property $n \cdot p = 0$ has been used. In the rest frame $\mathbf{p} = 0$, $n^\mu W_\mu$ becomes:

$$\begin{aligned} (n \cdot W)(\mathbf{p}=0) &= \frac{\hbar}{2}\gamma^5(n^i\gamma^i) p^0\gamma^0 = -\frac{\hbar}{2} mc\gamma^5\gamma^0\gamma^i n^i = -\frac{\hbar}{2} mc\gamma^5 \boldsymbol{\alpha}^i n^i \\ &= -mc\boldsymbol{\Sigma}\cdot\mathbf{n}, \end{aligned} \tag{10.200}$$

where we have used the property

$$\Sigma^i = \frac{\hbar}{2}\gamma^5\boldsymbol{\alpha}^i, \tag{10.201}$$

which can be verified using (10.194), (10.68) and (10.105). Thus we have found a Lorentz scalar quantity that in the rest frame reduces to $\mathbf{n}\cdot\boldsymbol{\Sigma}$:

$$O_n \equiv -\frac{1}{mc} n^\mu W_\mu \xrightarrow{\mathbf{p}=0} O_n^{(0)} = \mathbf{n}\cdot\boldsymbol{\Sigma}. \tag{10.202}$$

In the particular case of n pointing along the z-axis, $n = n_z = (0, 0, 0, 1)$, from (10.105) we find

$$O_{n_z}^{(0)} = -\frac{1}{mc} n^\mu W_\mu \Big|_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} \frac{\hbar}{2}\sigma_3 & 0 \\ 0 & \frac{\hbar}{2}\sigma_3 \end{pmatrix} = \Sigma_3. \tag{10.203}$$

Clearly, using the transformation property (10.196) of W^μ and the Lorentz invariance of the expression of O_n, in a generic frame $\mathcal{S}$ we find

$$O_n = -\frac{1}{mc} n^\mu W_\mu = S(\boldsymbol{\Lambda}_p) O_n^{(0)} S(\boldsymbol{\Lambda}_p)^{-1}, \tag{10.204}$$

that is if $u(\mathbf{0}, r)$, $v(\mathbf{0}, r)$ are eigenvectors on $\boldsymbol{\Sigma}\cdot\mathbf{n}$, $u(p, r)$, $v(p, r)$ are eigenvectors on O_n corresponding to the same eigenvalues, which is the content of (10.190) and (10.191).

We can define projectors $\mathcal{P}_r$ on eigenstates of O_n corresponding to the eigenvalues $\varepsilon_r\hbar/2 = \pm\hbar/2$:

$$\mathcal{P}_r \equiv \frac{1}{2}\left(\mathbf{1} + \varepsilon_r \frac{2}{\hbar} O_n\right) = \frac{1}{2}\left(\mathbf{1} + \varepsilon_r \frac{1}{mc}\gamma^5 \not{n}\not{p}\right). \tag{10.205}$$

In the rest frame the above projector reads:

$$\mathcal{P}_r^{(0)} \equiv \frac{1}{2}\left(\mathbf{1} + \varepsilon_r \gamma^5 n^i \boldsymbol{\alpha}_i\right) = \begin{pmatrix} \mathbf{1}_2 + \varepsilon_r \mathbf{n}\cdot\boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_2 + \varepsilon_r \mathbf{n}\cdot\boldsymbol{\sigma} \end{pmatrix}. \tag{10.206}$$

The matrices $\mathcal{P}_r$ project on both positive and negative energy solutions with the same spin component along $\mathbf{n}$. Let us now define two operators $\Lambda_{+,r}$, $\Lambda_{-,r}$ projecting on positive and negative solutions with a given spin component r, respectively:

$$\begin{aligned} &\Lambda_{+,r} u(p,s) = \delta_{rs} u(p,s); \quad \Lambda_{+,r} v(p,s) = 0, \\ &\Lambda_{-,r} u(p,s) = 0; \quad \Lambda_{-,r} v(p,s) = \delta_{rs} v(p,s). \end{aligned} \tag{10.207}$$

They have the following general form:

$$(\Lambda_{+,r})^\alpha{}_\beta = u^\alpha(p,r)\bar{u}_\beta(p,r); \quad (\Lambda_{-,r})^\alpha{}_\beta = -v^\alpha(p,r)\bar{v}_\beta(p,r), \tag{10.208}$$

as it follows from the orthogonality properties (10.168) and (10.169). To find the explicit expression of these matrices in terms of p and n, we notice that they are obtained by multiplying to the right and to the left the projectors $\mathcal{P}_r$ on the spin state r by the projectors $\Lambda_\pm$ on the positive and negative energy states:

$$\Lambda_{\pm,r} = \Lambda_\pm \mathcal{P}_r \Lambda_\pm = \Lambda_\pm \frac{1}{2}(\mathbf{1} \pm \varepsilon_r \gamma^5 \not{n}) = \pm\frac{1}{4mc}(\not{p} \pm mc)(\mathbf{1} \pm \varepsilon_r \gamma^5 \not{n}),$$

where we have used the property:

$$\left(\mathbf{1} + \varepsilon_r \frac{1}{mc}\gamma^5 \not{n}\not{p}\right)(\not{p} \pm mc) = (\not{p} \pm mc)(\mathbf{1} \pm \varepsilon_r \gamma^5 \not{n}), \tag{10.209}$$

which can be easily verified using the fact that $\not{p}$ and $\not{n}$ anticommute: $\not{n}\not{p} = -\not{p}\not{n}$.

10.7 Dirac Equation in an External Electromagnetic Field

We shall now study the coupling of the Dirac field to the electromagnetic field A_μ.

To this end, as we did for the complex scalar field in Sect. 10.2.1, we apply the *minimal coupling* prescription, namely we substitute in the free Dirac equation

$$p^\mu \to p^\mu + \frac{e}{c}A^\mu, \tag{10.210}$$

that is, in terms of the quantum operator

$$i\hbar\partial^\mu \to i\hbar\partial^\mu + \frac{e}{c}A^\mu. \tag{10.211}$$

In the convention which we adopt throughout the book, the electron has charge $e = -|e| < 0$. The coupled Dirac equation takes the following form:

$$\left[(i\hbar\partial_\mu + \frac{e}{c}A_\mu)\gamma^\mu - mc\right]\psi(x) = 0. \tag{10.212}$$

Using the *covariant derivative* introduced in (10.36), (10.212) takes the form

$$\left[i\hbar\gamma^\mu D_\mu - mc\right]\psi(x) = 0, \tag{10.213}$$

Just as in the case of the complex scalar field, the resulting equation is not invariant under gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\varphi(x), \tag{10.214}$$

unless we also apply to the Dirac wave function the following simultaneous phase transformation

$$\psi(x) \to \psi(x)e^{\frac{ie}{\hbar c}\varphi(x)}. \tag{10.215}$$

In connection with the discussion of the meaning of the charge-conjugation operation, it is instructive to see how the Dirac equation in the presence of an external electromagnetic field transforms under charge-conjugation. The equation for the charge-conjugate spinor $\psi^c = C\overline{\psi}^T = C\gamma^0\psi^*$ is easily derived from (10.212) and reads:

$$\left(\left(i\hbar\partial_\mu - \frac{e}{c}A_\mu\right)\gamma^\mu - mc\right)\psi^c(x) = 0. \tag{10.216}$$

We see that ψ and ψ^c describe particles with opposite charge. This justifies the statement given at the end of Sect. 10.6.2 that antiparticles have opposite charge with respect to the corresponding particles.[9] Let us now recast (10.212) in a Hamiltonian form. Solving with respect to the time derivative, we have:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-c\left(i\hbar\partial_i + \frac{e}{c}A_i\right)\boldsymbol{\alpha}^i + \boldsymbol{\beta}mc^2 - eA^0\right]\psi = H\psi, \tag{10.218}$$

where $H = H_{free} + H_{int}$, H_{free} being given by (10.53) and $H_{int} = -e(A_0 + A_i\boldsymbol{\alpha}^i)$. In order to study the physical implications of the minimal coupling it is convenient to study its non-relativistic limit. We proceed as in Sect. 10.4.1. We first redefine the Dirac field as in (10.73), so that the Dirac equation (10.218) takes the following form:

$$\left(i\hbar\frac{\partial}{\partial t} + mc^2\right)\psi' = \left[-c\left(i\hbar\partial_i + \frac{e}{c}A_i\right)\boldsymbol{\alpha}^i + \boldsymbol{\beta}mc^2 - eA^0\right]\psi'.$$

[9] We also observe that the Dirac equation is invariant under the transformations

$$\psi \to \psi^c, \quad A_\mu \to -A_\mu. \tag{10.217}$$

Next we decompose the field ψ' as in (10.71) (omitting prime symbols on φ and χ) and find:

$$\left(i\hbar\frac{\partial}{\partial t}+eA^0\right)\varphi=c\boldsymbol{\sigma}\cdot\left(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A}\right)\chi, \tag{10.219}$$

$$\left(i\hbar\frac{\partial}{\partial t}+eA^0+2mc^2\right)\chi=c\boldsymbol{\sigma}\cdot\left(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A}\right)\varphi. \tag{10.220}$$

As explained earlier, in the non-relativistic limit, we only keep on the left hand side of the second equation the term $2mc^2\chi$, since the rest energy mc^2 of the particle is much larger than the kinetic and potential energies, so that

$$\chi=\frac{1}{2mc}\boldsymbol{\sigma}\cdot\left(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A}\right)\varphi,$$

so that only the *large* upper component φ remains. Substituting the expression for χ into the first of (10.219) we obtain:

$$\left(i\hbar\frac{\partial}{\partial t}+eA^0\right)\varphi=\frac{1}{2m}\left[\boldsymbol{\sigma}\cdot\left(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A}\right)\right]^2\varphi. \tag{10.221}$$

To evaluate the right hand side we note that given two vectors **a, b** the following identity holds as a consequence of the Pauli matrix algebra:

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma})=\mathbf{a}\cdot\mathbf{b}+i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b}).$$

In our case

$$\mathbf{a}=\mathbf{b}=\left(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A}\right),$$

but the wedge product does not vanish, since $\boldsymbol{\nabla}$ and $\mathbf{A}$ do not commute. We find:

$$\begin{aligned}\left(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A}\right)\times\left(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A}\right)\varphi&=i\frac{e\hbar}{c}\left(-\mathbf{A}\times\boldsymbol{\nabla}+\boldsymbol{\nabla}\times\mathbf{A}\right)\varphi+i\frac{e\hbar}{c}\mathbf{A}\times\boldsymbol{\nabla}\varphi\\&=i\frac{e\hbar}{c}\mathbf{B}\varphi.\end{aligned} \tag{10.222}$$

Substituting in (10.221) we finally obtain:

$$i\hbar\frac{\partial\varphi}{\partial t}=\left[\frac{1}{2m}|i\hbar\boldsymbol{\nabla}+\frac{e}{c}\mathbf{A}|^2+eV-\frac{e}{mc}\mathbf{s}\cdot\mathbf{B}\right]\varphi\equiv H\varphi, \tag{10.223}$$

where we have defined, as usual, $\mathbf{s}\equiv\hbar\boldsymbol{\sigma}/2$, and written A_0 as $-V$, V being the electric potential. Equation (10.224) is called the *Pauli equation*. It differs from the Schroedinger equation of an electron interacting with the electromagnetic field by the presence in the Hamiltonian of the interaction term:

$$H_{magn} = -\frac{e}{mc}\mathbf{s}\cdot\mathbf{B} = -\boldsymbol{\mu}_s\cdot\mathbf{B}, \tag{10.224}$$

which has the form of the potential energy of a magnetic dipole in an external magnetic field with:

$$\boldsymbol{\mu}_s = \frac{e}{mc}\mathbf{s} = g\frac{e}{2mc}\mathbf{s}, \tag{10.225}$$

representing the electron *intrinsic magnetic moment*. The factor $g = 2$ is called the *g-factor* and the gyromagnetic ratio associated with the spin, defined as $|\boldsymbol{\mu}_s|/|\mathbf{s}|$, is $g|e|/(2mc)$. Recall that the magnetic moment associated with the orbital motion of a charge e reads

$$\boldsymbol{\mu}_{orbit} = \frac{e}{2mc}\mathbf{M}, \tag{10.226}$$

M being the orbital angular momentum. The gyromagnetic ratio $|\boldsymbol{\mu}_s|/|\mathbf{s}| = |e|/(mc)$ is *twice the one associated with the orbital angular momentum.* This result was found by Dirac in 1928.[10]

Finally we note that in the present non-relativistic approximation, taking into account that the small components χ can be neglected, the probability density $\psi^\dagger\psi = \varphi^\dagger\varphi + \chi^\dagger\chi$ reduces to $\varphi^\dagger\varphi$ as it must be the case for the Schroedinger equation.

Let us write the Lagrangian density for a fermion with charge e, coupled to the electromagnetic field:

$$\mathcal{L} = \bar{\psi}(x)\left(i\hbar c \not{D} - mc^2\right)\psi(x). \tag{10.227}$$

The reader can easily verify that the above Lagrangian yields (10.212), or, equivalently (10.213). Just as we did for the scalar field, we can write $\mathcal{L}$ as the sum of a part describing the free fermion, plus an interaction term $\mathcal{L}_I$, describing the coupling to the electromagnetic field:

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_0 + \mathcal{L}_I,\\
\mathcal{L}_0 &= \bar{\psi}(x)\left(i\hbar c\ \not{\partial} - mc^2\right)\psi(x),\\
\mathcal{L}_I &= A_\mu(x)J^\mu(x) = eA_\mu(x)\bar{\psi}(x)\gamma^\mu\psi(x),
\end{aligned} \tag{10.228}$$

where we have defined the electric current four vector J^μ as:

$$J^\mu(x) \equiv ej^\mu(x) = e\bar{\psi}(x)\gamma^\mu\psi(x). \tag{10.229}$$

In Sect. 10.4.2 we have shown that, by virtue of the Dirac equation, J^μ is a conserved current, namely that it is divergenceless: $\partial_\mu J^\mu = 0$.

[10] We recall that the Zeeman effect can only be explained if $g = 2$. We see that this value is correctly predicted by the Dirac relativistic equation in the non-relativistic limit.

10.8 Parity Transformation and Bilinear Forms

It is important to observe that the standard representation of the γ-matrices given in (10.4.1) is by no means unique. Any other representation preserving the basic anticommutation rules works exactly the same way. It is only a matter of convenience to use one or the another. In particular the expression (10.97) of the Lorentz generators $\Sigma^{\mu\nu}$ in terms of γ^μ-matrices is representation-independent.

In this section we introduce a different representation, called the *Weyl representation*, defined as follows:

$$\gamma^0 = \begin{pmatrix} \mathbf{0} & \mathbf{1}_2 \\ \mathbf{1}_2 & \mathbf{0} \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} \mathbf{0} & -\sigma^i \\ \sigma^i & \mathbf{0} \end{pmatrix}; \quad i = 1, 2, 3. \tag{10.230}$$

It is immediate to verify that the basic anticommutation rules (10.61) are satisfied. Defining

$$\sigma^\mu = (\mathbf{1}_2, -\sigma^i); \quad \bar{\sigma}^\mu = (\mathbf{1}_2, \sigma^i), \tag{10.231}$$

equation (10.230) can be given the compact form

$$\gamma^\mu = \begin{pmatrix} \mathbf{0} & \sigma^\mu \\ \bar{\sigma}^\mu & \mathbf{0} \end{pmatrix}. \tag{10.232}$$

The standard (Pauli) and the Weyl representations are related by a unitary change of basis:

$$\gamma^\mu_{Pauli} = U^\dagger \gamma^\mu_{Weyl} U.$$

Decomposing as usual the spinor ψ into two-dimensional spinors ξ e ζ:

$$\psi = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \tag{10.233}$$

one can show that, in the Weyl representation, the *proper* Lorentz transformations act separately on the two spinors, without mixing them. As we are going to show below, this means that the four-dimensional spinor representation, *irreducible* with respect to the full Lorentz group $O(1, 3)$ becomes *reducible* into two two-dimensional representations under the subgroup of the *proper* Lorentz group $SO(1, 3)$.

To show this we observe that since infinitesimal transformations in the spinor representation of the Lorentz group are, by definition, connected with continuity to the identity, they ought to have unit determinant, and therefore they can only belong to the subgroup of proper Lorentz transformations $SO(1, 3)$.

We can compute, in the Weyl basis, the matrix form of the $\Sigma^{\mu\nu}$ generators:

$$\Sigma^{\mu\nu} = -\frac{\hbar}{2}\sigma^{\mu\nu} = -\frac{i\hbar}{4}\left[\gamma^\mu, \gamma^\nu\right] \tag{10.234}$$

$$= -\frac{i\hbar}{4}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix} \quad (10.235)$$

and restricting $\mu\nu$ to space indices we have

$$\Sigma_i = -\frac{1}{2}\epsilon_{ijk}\Sigma^{jk} = \frac{\hbar}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}. \quad (10.236)$$

The generators Σ_i of rotations $S(\mathbf{\Lambda}_R)$ have the same form as in the Pauli representation. the corresponding finite transformation will therefore be implemented on spinors by the same matrix $S(\mathbf{\Lambda}_R)$ in (10.108).

Moreover from (10.100) the spinor representation of the infinitesimal *boost* generators J^{0i}, are also given in terms of a block diagonal matrix

$$\Sigma^{0i} = -i\hbar K_i = -i\hbar\frac{\boldsymbol{\alpha}^i}{2} = -\frac{i\hbar}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}. \quad (10.237)$$

It follows that if we use the decomposition (10.233) a *proper* Lorentz transformation can never mix the upper and lower components of the Dirac spinor ψ. The explicit finite form of the *proper* Lorentz transformations in the spinor representation can be found by exponentiation of the generators, following the method explained in Chap. 7.

A generic proper Lorentz transformation can be written as the product of a rotation and a boost transformation, as in (10.112). The rotation part was given in (10.108), while the boost part $S(\mathbf{\Lambda}_B)$ was given in (10.116) in terms of the matrices $\boldsymbol{\alpha}^i$, whose matrix representation now, in the Weyl basis, is different. One finds that under $\mathbf{\Lambda}_{R,B}$ the two two-spinors ξ, ζ transform as follows:

$$\xi \xrightarrow{\mathbf{\Lambda}_R} \left[\cos\frac{\theta}{2} + i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\theta}}\sin\frac{\theta}{2}\right]\xi; \quad \zeta \xrightarrow{\mathbf{\Lambda}_R} \left[\cos\frac{\theta}{2} + i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\theta}}\sin\frac{\theta}{2}\right]\zeta,$$
$$\xi \xrightarrow{\mathbf{\Lambda}_B} \left[\cosh\frac{\lambda}{2} + \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\lambda}}\sinh\frac{\lambda}{2}\right]\xi; \quad \zeta \xrightarrow{\mathbf{\Lambda}_B} \left[\cosh\frac{\lambda}{2} - \boldsymbol{\sigma}\cdot\hat{\boldsymbol{\lambda}}\sinh\frac{\lambda}{2}\right]\zeta,$$

where $\theta \equiv |\boldsymbol{\theta}|$; $\lambda \equiv |\boldsymbol{\lambda}|$; $\hat{\boldsymbol{\lambda}} = \frac{\boldsymbol{\lambda}}{|\boldsymbol{\lambda}|}$; $\boldsymbol{\theta} = \frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|}$.

The above results refer to proper Lorentz transformations, that is they exclude transformations with negative determinant: $\det\mathbf{\Lambda} = -1$. Let us now consider Lorentz transformations with $\det\mathbf{\Lambda} = -1$. Keeping $\Lambda^0_{\ 0} > 0$, the typical transformation with $\det\mathbf{\Lambda} = -1$ is the *parity transformation* $\mathbf{\Lambda}_P \in \mathrm{O}(1,3)$ defined by the following *improper* Lorentz matrix:

$$(\Lambda_P)^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (10.238)$$

On the space-time coordinates x^μ it acts as follows :

$$x^0 \to x^0; \quad \mathbf{x} \to -\mathbf{x}, \tag{10.239}$$

that is it corresponds to a change of the orientation of the three coordinate axes.

We now show that $\mathbf{\Lambda}_P$ acts on spinors as follows:

$$S(\mathbf{\Lambda}_P) = \eta_P \gamma^0, \tag{10.240}$$

where $\eta_P = \pm 1$.

We may indeed verify that

$$S(\mathbf{\Lambda}_P)^{-1} \gamma^\mu S(\mathbf{\Lambda}_P) = \Lambda_P{}^\mu_\nu \gamma^\nu.$$

which generalizes the general formula (10.89) to the parity transformation. The above property is readily proven, using (10.240):

$$\begin{aligned} S(\mathbf{\Lambda}_P)^{-1} \gamma^0 S(\mathbf{\Lambda}_P) &= \gamma^0 = \mathbf{\Lambda}_P{}^0{}_0 \gamma^0, \\ S(\mathbf{\Lambda}_P)^{-1} \gamma^i S(\mathbf{\Lambda}_P) &= -\gamma^i = \mathbf{\Lambda}_P{}^i{}_j \gamma^j. \end{aligned} \tag{10.241}$$

The action of a parity transformation on a Dirac field $\psi(x)$ is therefore:

$$\psi(x) \xrightarrow{P} \eta_P \gamma^0 \psi(x^0, -\mathbf{x}). \tag{10.242}$$

If we take into account that in the Weyl representation the γ-matrices are given by (10.230) and are off-diagonal, we see that the parity transformation $\mathbf{\Lambda}_P$ transforms ξ and ζ into one another:

$$\begin{cases} \xi \to \eta_P \chi, \\ \chi \to \eta_P \xi. \end{cases} \tag{10.243}$$

This result shows that while for *proper* Lorentz transformations the representation of the Lorentz group is reducible since it acts separately on the two spinor components, if we consider the full the Lorentz group, including also improper transformations like parity, the representation becomes irreducible and we are bound to use four-dimensional spinors.

Let us now write the Dirac equation in this new basis. On momentum eigenstates $w(p)e^{-\frac{i}{\hbar} p \cdot x}$ it reads:

$$(\not{p} - mc)w(p) = 0 \Rightarrow \begin{cases} (p^0 - \mathbf{p} \cdot \boldsymbol{\sigma})\xi = mc\zeta, \\ (p^0 + \mathbf{p} \cdot \boldsymbol{\sigma})\zeta = mc\xi, \end{cases} \tag{10.244}$$

where we have written $w = (\xi, \zeta)$. For massless spinors $m = 0$ the above equations decouple:

$$(p^0 - \mathbf{p} \cdot \boldsymbol{\sigma})\xi = 0; \quad (p^0 + \mathbf{p} \cdot \boldsymbol{\sigma})\zeta = 0, \tag{10.245}$$

which will have solutions for $p^0 > 0$ and $p^0 < 0$. The above equations fix the helicity Γ of the solution which, as we know, is a conserved quantity and labels the internal degrees of freedom of a massless particle.[11] On the two two-spinors ξ, ζ, the helicity is indeed $\Gamma = \hbar \mathbf{p} \cdot \boldsymbol{\sigma}/(2|\mathbf{p}|) = \hbar \mathbf{p} \cdot \boldsymbol{\sigma}/(2p^0)$: It is positive for negative energy solutions ζ and positive energy solutions ξ, while it is negative for positive energy solutions ζ and negative energy solutions ξ.

In nature there are three spin 1/2 particles, called neutrinos and denoted by ν_e, ν_μ, ν_τ, which, until recently, were believed to be massless.

In next chapter we shall be dealing with the other improper Lorentz transformation besides parity, which is *time-reversal*.

10.8.1 Bilinear Forms

Let us now consider the matrix γ^5, introduced in (10.194). Its explicit form in the Weyl representation is

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \begin{pmatrix} \mathbf{1}_2 & \mathbf{0} \\ \mathbf{0} & -\mathbf{1}_2 \end{pmatrix}. \tag{10.246}$$

Let us investigate the transformation properties of γ^5 under a general Lorentz transformation:

$$\begin{aligned} S^{-1}(\boldsymbol{\Lambda})^{-1}\gamma^5 S(\boldsymbol{\Lambda}) &= \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma} S^{-1}\gamma^\mu S S^{-1}\gamma^\nu S S^{-1}\gamma^\rho S S^{-1}\gamma^\sigma S \\ &= \frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\Lambda^\rho{}_{\rho'}\Lambda^\sigma{}_{\sigma'}\gamma^{\mu'}\gamma^{\nu'}\gamma^{\rho'}\gamma^{\sigma'} \\ &= \det(\boldsymbol{\Lambda})\frac{i}{4}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \\ &= \det(\boldsymbol{\Lambda})\gamma^5. \end{aligned} \tag{10.247}$$

In particular under a *parity transformation*, being $\det \boldsymbol{\Lambda}_P = -1$ we have:

$$S(\boldsymbol{\Lambda}_P)^{-1}\gamma^5 S(\boldsymbol{\Lambda}_P) = -\gamma^5, \tag{10.248}$$

that is, it transforms as a *pseudoscalar*. By the same token we can show that:

$$S(\boldsymbol{\Lambda})^{-1}\gamma^5\gamma^\mu S(\boldsymbol{\Lambda}) = \det(\boldsymbol{\Lambda})\Lambda^\mu{}_\nu(\gamma^5\gamma^\nu). \tag{10.249}$$

[11] Recall that helicity is invariant under proper Lorentz transformations and labels irreducible representations of SO(1, 3) with $m = 0$.

so that $\gamma^5\gamma^\mu$ transforms as an *pseudo-vector*, that is as an ordinary vector under proper Lorentz transformations, and with an additional minus sign under parity. Defining

$$\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu],$$

we verify that $\gamma^{\mu\nu}$ transforms an *antisymmetric tensor of rank two*:

$$S(\mathbf{\Lambda})^{-1}\gamma^{\mu\nu}S(\mathbf{\Lambda}) = \frac{1}{2}\left[S^{-1}\gamma^\mu S, S^{-1}\gamma^\nu S\right] = \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\gamma^{\rho\sigma} \tag{10.250}$$

while $\gamma_5\gamma^{\mu\nu}$ transforms like a pseudo- (or axial-) tensor, that is with an additional minus sign under parity as it follows from (10.248):

$$S(\mathbf{\Lambda})^{-1}\gamma_5\gamma^{\mu\nu}S(\mathbf{\Lambda}) = \det(\mathbf{\Lambda})\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma\gamma^5\gamma^{\mu\nu}. \tag{10.251}$$

These properties allow us to construct bilinear forms in the spinor fields ψ which have definite transformation under the full Lorentz group.

Indeed if we consider a general bilinear form of the type:

$$\bar{\psi}(x)\gamma^{\mu_1\cdots\mu_k}\psi(x) \tag{10.252}$$

as shown in Appendix G the independent bilinears are:

$$\bar{\psi}(x)\psi(x); \quad \bar{\psi}(x)\gamma^\mu\psi(x); \quad \bar{\psi}(x)\gamma^{\mu\nu}\psi(x); \quad \bar{\psi}(x)\gamma^5\psi(x); \quad \bar{\psi}(x)\gamma^5\gamma^\mu\psi(x). \tag{10.253}$$

To exhibit their transformation properties we perform the transformation

$$\psi'(x') = S\psi(x) \to \bar{\psi}'(x') = \overline{S\psi(x)} = \psi^\dagger(x)S^\dagger\gamma^0 \tag{10.254}$$

and use the relation (10.92) of Sect. 9.3.3, namely

$$\gamma^0 S^\dagger \gamma^0 = S^{-1} \tag{10.255}$$

Using (10.247) and (10.248) it is easy to show that $\bar{\psi}(x)\psi(x)$ is a scalar field while $\bar{\psi}(x)\gamma^5\psi(x)$ is a pseudoscalar, i.e. under parity they transform as follows:

$$\bar{\psi}(x)\psi(x) \to \bar{\psi}'(x')\psi'(x'); \quad \bar{\psi}(x)\gamma^5\psi(x) \to -\bar{\psi}'(x')\gamma^5\psi'(x'). \tag{10.256}$$

By the same token, and using (10.250) and (10.251) as well, we find analogous transformation properties for the remaining fermion bilinears. The result is summarized in the following table:

Bilinear	P-transformed	Kind
$\bar{\psi}(x)\psi(x)$	$\bar{\psi}(x_P)\psi(x_P)$	Scalar field
$\bar{\psi}(x)\gamma^5\psi(x)$	$-\bar{\psi}(x_P)\gamma^5\psi(x_P)$	Pseudo-scalar field
$\bar{\psi}(x)\gamma^\mu\psi(x)$	$\eta_{\mu\mu}\bar{\psi}(x_P)\gamma^\mu\psi(x_P)$	Vector field
$\bar{\psi}(x)\gamma^5\gamma^\mu\psi(x)$	$-\eta_{\mu\mu}\bar{\psi}(x_P)\gamma^5\gamma^\mu\psi(x_P)$	Axial-vector field
$\bar{\psi}(x)\gamma^{\mu\nu}\psi(x)$	$\eta_{\mu\mu}\eta_{\nu\nu}\bar{\psi}(x_P)\gamma^{\mu\nu}\psi(x_P)$	(Antisymmetric) tensor field

where, in the second column, there is no summation over the μ and ν indices, and $x_P \equiv (x_P^\mu) = (ct, -\mathbf{x})$.

Reference

For further readings see Refs. [3], [8] (Vol. 4), [9], [13]