8.5 Lagrangian and Hamiltonian Formalism in Field Theories

Our discussion has been confined so far to mechanical systems with a *finite* number of degrees of freedom, $q_1(t), \ldots, q_n(t)$.

This has been propaedeutic to our principal interest, namely the description of *continuous* systems, hereafter called *fields*. A well known example of field is the *electromagnetic field* whose description is given in terms of the four-potential $A_\mu(\mathbf{x}, t)$; that means that, at any instant t, its configuration is defined by assigning, for each component μ, the value of $A_\mu(\mathbf{x}, t)$ *at each point* $\mathbf{x}$ *in space*.

In this case we have a *continuous infinity of canonical coordinates* $q_i(t) = A_\mu(\mathbf{x}, t)$, labeled by the three coordinates $\mathbf{x}$ for the space-point and the index μ.[14] Other examples of fields are the continuous matter fields like fluids, elastic media, etc.

Quite generally we may view a continuous system as the limit of a mechanical system described by a finite number of degrees of freedom q_i (*discrete system*), by letting i become the continuous index $\mathbf{x}$. As a consequence every sum Σ_i over the discrete label i will be replaced by an integration on $d^3\mathbf{x}$ over a spatial domain V, usually the whole three-dimensional space[15]:

$$\sum_i \to \int_V d^3\mathbf{x}.$$

In the following we shall consider fields $\varphi^\alpha(\mathbf{x}, t)$ carrying an (internal) index α, where α labels the components of a "*vector* $\varphi \equiv (\varphi^\alpha)$" on which a representation of a group G acts. If α has just one value, it will be omitted and we speak of a *scalar field*. In *relativistic field theories*, the group G will often be the Lorentz group $\mathrm{O}(1, 3)$ so that the index will label a basis of the carrier of a representation of the Lorentz group.[16] For example, in the case of the electromagnetic field, the role of α is played by the index μ pertaining to the four-dimensional fundamental representation of $\mathrm{SO}(1, 3)$.

8.5.1 *Functional Derivative*

When we think of fields as a continuous limit of discrete systems, the corresponding Lagrangian obtained in the limit, $L(\varphi^\alpha, \partial_t\varphi^\alpha, t)$, will depend, at a certain instant t,

[14] More precisely, since $\mathbf{x} \equiv (x^1, x^2, x^3)$, we have a triple infinity of Lagrangian coordinates $q_i(t)$ for each value of the index $\mu = 0, 1, 2, 3$. The three components of $\mathbf{x}$ and the index μ play the role of the index i of the discrete case.

[15] Actually in our treatment of a discrete number of degrees of freedom, we have often omitted the symbol Σ when there are repeated indices.

[16] Somewhat improperly, by the word representation people often refer to the carrier space V_p of a representation. We shall also do this to simplify the exposition and thus talk about a *basis of a representation* when referring to a basis of the corresponding carrier space.

on the values of the fields $\varphi^\alpha(\mathbf{x}, t)$ and $\partial_t\varphi^\alpha(\mathbf{x}, t)$ *at every point in the domain V of the three-dimensional space.* We say in this case that the Lagrangian is a *functional* of $\varphi^\alpha(\mathbf{x}, t)$ and $\partial_t\varphi^\alpha(\mathbf{x}, t)$, viewed as functions of $\mathbf{x}$. It will be convenient in the following to denote by $\varphi^\alpha(t)$ the function $\varphi^\alpha(\mathbf{x}, t)$ of the point $\mathbf{x}$ in space at a given time t, and by $\dot{\varphi}^\alpha(t)$ its time derivative $\dot{\varphi}^\alpha(\mathbf{x}, t) \equiv \partial_t\varphi^\alpha(\mathbf{x}, t)$. We shall presently explore some property of functionals. Let us consider a functional $F[\varphi]$, and perform an independent variation of $\varphi(\mathbf{x})$, at each space point $\mathbf{x}$. The corresponding variation of $F[\varphi]$ will be:

$$\delta F[\varphi] \equiv F[\varphi + \delta\varphi] - F[\varphi] = \int \frac{\delta F[\varphi]}{\delta\varphi(\mathbf{x})}\delta\varphi(\mathbf{x})d^3\mathbf{x}, \tag{8.103}$$

where *by definition*, $\frac{\delta F[\varphi]}{\delta\varphi(\mathbf{x})}$ is the *functional derivative* of $F[\varphi]$ with respect to φ at the point $\mathbf{x}$. Here we have suppressed the possible dependence on time of φ and of the functional F either explicitly or through φ: $\varphi = \varphi(\mathbf{x}, t)$, $F = F[\varphi(t), t]$.

From its definition it is easy to verify that the functional derivation enjoys the same properties as the ordinary one, namely it is a linear operator, vanishes on constants and satisfies the Leibnitz rule.

When the functional depends on more than a single function, its definition can be extended correspondingly, as for ordinary derivatives. Of particular relevance for us is the additional dependence of F on the time derivative $\partial_t\varphi(\mathbf{x}, t)$ of $\varphi(\mathbf{x}, t)$. Moreover we may consider a set of fields φ^α labeled by the index α pertaining to a given representation of a group G. This is the case of the Lagrangian $F = L(\varphi^\alpha(t), \dot{\varphi}^\alpha(t), t)$, where we recall once again that, in writing $\varphi(t), \dot{\varphi}(t)$ among the arguments of the Lagrangian, we mean that L depends on the values $\varphi(\mathbf{x}, t), \dot{\varphi}(\mathbf{x}, t)$ of these fields in *every point* $\mathbf{x}$ in space at a given time t. Applying the definition (8.103) to the two functions $\varphi^\alpha(t)$ and $\dot{\varphi}^\alpha(t)$ we have:

$$\delta L(\varphi^\alpha(t), \dot{\varphi}^\alpha(t), t) = \int d^3\mathbf{x}\left[\frac{\delta L}{\delta\varphi^\alpha(\mathbf{x}, t)}\delta\varphi^\alpha(\mathbf{x}, t) + \frac{\delta L}{\delta\dot{\varphi}^\alpha(\mathbf{x}, t)}\delta\dot{\varphi}^\alpha(\mathbf{x}, t)\right]. \tag{8.104}$$

Note that the Lagrangian depends on t either through φ^α and $\dot{\varphi}^\alpha$ or explicitly. The Lagrangian, as a *functional* with respect to the space-dependence of the fields, can be thought of as the continuous limit of a function of infinitely many discrete variables:

$$L(\varphi_i(t), \dot{\varphi}_i(t), t) \xrightarrow{i\to\mathbf{x}} L(\varphi(t), \dot{\varphi}(t), t).$$

Here and in the following we shall often omit the index α if not essential to our considerations. Correspondingly, we can show that the *functional derivative* defined above can be thought of as a suitable continuous limit of the ordinary derivative with respect to a discrete set of degrees of freedom q_i, described by a Lagrangian $L(q_i, \dot{q}_i)$. Let us indeed regard the values of $\varphi(\mathbf{x}, t)$ at each point $\mathbf{x}$ as independent canonical coordinates. To deal with a continuous infinity of canonical coordinates, we divide the 3-space into tiny cells of volume δV^i. Let $\varphi_i(t)$ be the mean value of $\varphi(\mathbf{x}, t)$ inside the ith cell and $L(t) = L(\varphi_i(t), \dot{\varphi}_i(t), t)$ be the Lagrangian, depending on the values $\varphi_i(t), \dot{\varphi}_i(t)$ of the field and its dime derivative in every cell. The variation $\delta L(\varphi_i, \dot{\varphi}_i)$ can be written as

$$\delta L(\varphi_i(t), \dot{\varphi}_i(t), t) = \sum_i \left(\frac{\partial L}{\partial \varphi^i} \delta\varphi_i + \frac{\partial L}{\partial \dot{\varphi}^i} \delta\dot{\varphi}_i \right)$$
$$= \sum_i \frac{1}{\delta V^i} \left(\frac{\partial L}{\partial \varphi^i} \delta\varphi_i + \frac{\partial L}{\partial \dot{\varphi}^i} \delta\dot{\varphi} \right) \delta V^i, \tag{8.105}$$

If we compare this expression with (8.104), in the continuum limit one can make the following identification:

$$\frac{\delta L}{\delta\varphi(\mathbf{x}, t)} \equiv \lim_{\delta V^i \to 0} \frac{1}{\delta V^i} \frac{\partial L}{\partial \varphi^i}, \tag{8.106}$$
$$\frac{\delta L}{\delta\dot{\varphi}(\mathbf{x}, t)} \equiv \lim_{\delta V^i \to 0} \frac{1}{\delta V^i} \frac{\partial L}{\partial (\dot{\varphi}^i)},$$

where $\mathbf{x}$ is in the ith cell. In the limit $\delta V_i \to 0$ we can set $\delta V_i \equiv d^3\mathbf{x}$. Thus the functional derivative $\delta L(t)/\delta\varphi(\mathbf{x}, t)$ is essentially proportional to the derivative of L with respect to the value of φ at the point $\mathbf{x}$. Since in the discretized notation the action principle leads to the equations of motion:

$$\frac{\partial L(t)}{\partial \varphi_i} - \partial_t \frac{\partial L(t)}{\partial \dot{\varphi}_i(t)} = 0 \tag{8.107}$$

in the continuum limit the Euler–Lagrange equations become:

$$\frac{\delta L}{\delta\varphi^\alpha(\mathbf{x}, t)} - \partial_t \frac{\delta L}{\delta\dot{\varphi}^\alpha(\mathbf{x}, t)} = 0. \tag{8.108}$$

where we have reintroduced the index α of the general case.

In the discretized notation we shall assume the Lagrangian L, which depends on the values of the fields and their time derivatives in every cell, to be the sum of quantities $\mathcal{L}_i$ defined in each cell: $\mathcal{L}_i$ depends on the values of the field $\varphi_i^\alpha(t)$, its gradient $\nabla\varphi_i^\alpha$ and its time derivative $\dot{\varphi}_i^\alpha(t)$ in the ith cell only:

$$L(\varphi_i^\alpha(t), \dot{\varphi}_i^\alpha(t), t) = \sum_i \mathcal{L}_i(\varphi_i^\alpha(t), \nabla\varphi_i^\alpha(t), \dot{\varphi}_i^\alpha(t), t). \tag{8.109}$$

Multiplying and dividing the right hand side by δV_i and taking the continuum limit $\delta V_i \to d^3\mathbf{x}$, the above equality becomes

$$L(\varphi^\alpha(t), \dot{\varphi}^\alpha(t), t) = \int_V d^3\mathbf{x} \mathcal{L}(\varphi^\alpha(x), \nabla\varphi^\alpha(x), \dot{\varphi}^\alpha(x); \mathbf{x}, t), \tag{8.110}$$

where $x \equiv (x^\mu) = (ct, \mathbf{x})$ and we have defined the *Lagrangian density* $\mathcal{L}$ as

$$\mathcal{L}(\varphi^\alpha(x), \nabla\varphi^\alpha(x), \dot{\varphi}^\alpha(x); \mathbf{x}, t) \equiv \lim_{\delta V_i \to 0} \frac{1}{\delta V_i} \mathcal{L}_i(\varphi_i^\alpha(t), \nabla\varphi_i^\alpha(t), \dot{\varphi}_i^\alpha(t), t).$$

Just as $\mathcal{L}_i$ depends, at a time t, on the dynamic variables referred to the i-th cell only, $\mathcal{L}$ is a *local* quantity in Minkowski space in that it depends on both $\mathbf{x}$ and t. We note the appearance in $\mathcal{L}(x)$ of the space derivatives $\nabla\varphi^\alpha(\mathbf{x}, t)$. This follows from the fact that in order to have an *action* which is a *scalar* under Lorentz transformations, $\mathcal{L}$ itself must be a Lorentz scalar. Since Lorentz transformations will in general shuffle time and space derivatives, $\mathcal{L}$ should then depend on all of them. The action, in terms of the Lagrangian density, will read:

$$\mathcal{S}[\varphi^\alpha; t_1, t_2] = \int_{t_1}^{t_2} L(t)dt = \int dt d^3\mathbf{x}\mathcal{L}(x) = \frac{1}{c}\int_{D_4} d^4x\mathcal{L}(x), \tag{8.111}$$

where D_4 is a space–time domain: An event $x \equiv (x^\mu)$ in D_4 occurs at a time t between t_1 and t_2 and at a point $\mathbf{x}$ in the volume V. In formulas we will write $D_4 \equiv [t_1, t_2] \times V \subset M_4$. Since $\mathcal{S}$ does not depend only on the time interval $[t_1, t_2]$ but also on the volume V in which the values of the fields and their derivatives are considered, we will write $\mathcal{S} \equiv \mathcal{S}[\varphi^\alpha; D_4]$. The boundary of D_4, to be denoted by ∂D_4, consists of all the events occurring either at $t = t_1$ or at $t = t_2$, and of events occurring at a generic $t \in [t_1, t_2]$ in a point $\mathbf{x}$ belonging to the surface S_V which encloses the volume V: $\mathbf{x} \in S_V \equiv \partial V$. The measure of integration $d^4x \equiv dx^0dx^1dx^2dx^3 = cdtd^3\mathbf{x}$ is invariant under Lorentz transformations $\mathbf{\Lambda} = (\Lambda^\mu{}_\nu)$, since the absolute value $|\det(\mathbf{\Lambda})|$ of the determinant of the corresponding Jacobian matrix $\mathbf{\Lambda}$, is equal to one

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu \Rightarrow d^4x \longrightarrow d^4x' = |\det(\mathbf{\Lambda})|d^4x = d^4x. \tag{8.112}$$

It follows that in order to have a *scalar* Lagrangian density $\mathcal{L}$ must have the same dependence on $\nabla\varphi^\alpha(\mathbf{x}, t)$ as on $\dot{\varphi}^\alpha(\mathbf{x}, t)$, that is it must actually depend on the four-vector $\partial_\mu\varphi^\alpha(\mathbf{x}, t)$. Moreover, being a scalar, it must depend on the fields and their derivatives $\partial_\mu\varphi^\alpha(\mathbf{x}, t)$ only through invariants constructed out of them. For the same reason it cannot depend on t only, but, in general, on all the space–time coordinates x^μ.

Let us now consider arbitrary infinitesimal variations of the field $\varphi^\alpha(x)$ which vanish at the boundary ∂D_4 of D_4: $\delta\varphi^\alpha(x) \equiv 0$ if $x \in \partial D_4$. The corresponding variation of L can be computed by using (8.110):

$$\begin{aligned}\delta L &= \int d^3\mathbf{x}\left[\frac{\partial\mathcal{L}(\mathbf{x}, t)}{\partial\varphi^\alpha(\mathbf{x}, t)}\delta\varphi^\alpha(\mathbf{x}, t) + \frac{\partial\mathcal{L}(\mathbf{x}, t)}{\partial\partial_i\varphi^\alpha(\mathbf{x}, t)}\delta\partial_i\varphi^\alpha(\mathbf{x}, t)\right.\\ &\quad\left. + \frac{\partial\mathcal{L}(\mathbf{x}, t)}{\partial(\dot{\varphi}^\alpha(\mathbf{x}, t))}\delta\dot{\varphi}^\alpha(\mathbf{x}, t)\right]\\ &= \int d^3\mathbf{x}\left\{\left[\frac{\partial\mathcal{L}(\mathbf{x}, t)}{\partial\varphi^\alpha(\mathbf{x}, t)} - \partial_i\frac{\partial\mathcal{L}(\mathbf{x}, t)}{\partial\partial_i\varphi^\alpha(\mathbf{x}, t)}\right]\delta\varphi^\alpha(\mathbf{x}, t) - \frac{\partial\mathcal{L}(\mathbf{x}, t)}{\partial\dot{\varphi}^\alpha(\mathbf{x}, t)}\delta\dot{\varphi}^\alpha(\mathbf{x}, t)\right\},\end{aligned} \tag{8.113}$$

where we have written $\nabla \equiv (\partial_i)_{i=1,2,3}$, used the property that $\delta\partial_i\varphi^\alpha = \partial_i\delta\varphi^\alpha$ and integrated the second term within the integral by parts, dropping the surface term, being $\delta\varphi^\alpha(x) = 0$ for $\mathbf{x} \in S_V \equiv \partial V$.

Taking into account that the quantity inside the curly brackets defines the functional derivative of L, by comparison with (8.108) we find:

$$\frac{\delta L}{\delta\varphi^\alpha(x)} = \left[\frac{\partial\mathcal{L}(x)}{\partial\varphi^\alpha(x)} - \partial_i\frac{\partial\mathcal{L}(x)}{\partial\partial_i\varphi^\alpha(x)}\right],$$
$$\frac{\delta L}{\delta\dot{\varphi}^\alpha(x)} = \frac{\partial\mathcal{L}(x)}{\partial\dot{\varphi}^\alpha(x)}. \tag{8.114}$$

It is important to note that, using the *Lagrangian density* instead of the Lagrangian, the derivatives of $\mathcal{L}(\mathbf{x}, t)$ with respect to the fields in $(\mathbf{x}, t)$ are now *ordinary derivatives*, since they are computed at a particular point $\mathbf{x}$. Using the equalities (8.114) the Euler–Lagrange equations (8.108) take the following form:

$$\partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\varphi^\alpha)} = \frac{\partial\mathcal{L}}{\partial\varphi^\alpha} - \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\varphi^\alpha)}, \tag{8.115}$$

or, using a Lorentz covariant notation:

$$\frac{\partial\mathcal{L}}{\partial\varphi^\alpha} - \partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\right) = 0. \tag{8.116}$$

8.5.2 The Hamilton Principle of Stationary Action

In the previous paragraph the equations of motion for fields have been derived using the definition of functional derivative and performing the continuous limit of the Euler–Lagrange equations for a discrete system.

Actually (8.116), can also be derived directly from the Hamilton principle of stationary action, considering the action $\mathcal{S}$ as a functional of the fields φ^α and depending on the space-time domain D_4 on which they are defined:

$$\mathcal{S}\left[\varphi^\alpha; D_4\right] = \frac{1}{c}\int_{D_4} d^4x\,\mathcal{L}(\varphi^\alpha, \partial_\mu\varphi^\alpha, x^\mu). \tag{8.117}$$

Here $d^4x \equiv dx^0 d^3\mathbf{x} = c\,dt\,d^3\mathbf{x}$ is the volume element in the Minkowski space M_4, and the integration domain D_4 was defined as $[t_1, t_2] \times V \subset M_4$.

We can now generalize the *Hamilton principle of stationary action* to systems described by fields, namely systems exhibiting a continuous infinity of degrees of freedom. It states that:

The time evolution of the field configuration describing the system is obtained by extremizing the action with respect to arbitrary variations of the fields $\delta\varphi^\alpha$ which vanish at the boundary ∂D_4 of the space–time domain D_4.

Precisely, we require the action $\mathcal{S}$ to be stationary with respect to $\delta\varphi^\alpha$, that is to satisfy

$$\delta\mathcal{S} = 0,$$

under arbitrary variations of φ^α at each point $\mathbf{x}$ and at each instant t:

$$\varphi^\alpha(x) \rightarrow \varphi^\alpha(x) + \delta\varphi^\alpha(x),$$

provided:

$$\delta\varphi^\alpha(x) = 0 \quad \forall x^\mu \in \partial D_4. \tag{8.118}$$

Let us apply this principle to the action (8.117). We have:

$$\delta\mathcal{S} = \frac{1}{c}\int_{D_4} d^4x \left(\frac{\partial\mathcal{L}}{\partial\varphi^\alpha}\delta\varphi^\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\delta(\partial_\mu\varphi^\alpha)\right). \tag{8.119}$$

Now use the property $\delta(\partial_\mu\varphi^\alpha) = \partial_\mu(\delta\varphi^\alpha)$, and integrate by parts the second term in the integral:

$$\begin{aligned}\int_{D_4} d^4x \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\partial_\mu\delta\varphi^\alpha &= \int_{D_4} d^4x\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\delta\varphi^\alpha\right)\\ &\quad -\int_{D_4} d^4x\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\right)\delta\varphi^\alpha = \int_{\partial D_4} d^4\sigma_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\right)\delta\varphi^\alpha\\ &\quad -\int_{D_4} d^4x\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\right)\delta\varphi^\alpha = -\int_{D_4} d^4x\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\right)\delta\varphi^\alpha,\end{aligned}$$

where we have applied the four-dimensional version of the divergence theorem by expressing the integral of a four-divergence over D_4 as an integral (boundary integral) of the four-vector $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\delta\varphi^\alpha$ over the three-dimensional domain ∂D_4 which encloses D_4. We have used the notation $d^3\sigma_\mu \equiv d^3\sigma n_\mu$, $d^3\sigma$ being an element of ∂D_4 to which the unit norm vector n_μ is normal, see Fig. 8.1. As for the last equality we have used (8.118) which implies the vanishing of the boundary integral.[17] Thus the partial integration finally gives:

$$\delta\mathcal{S} = \frac{1}{c}\int_{D_4} d^4x \left[\frac{\partial\mathcal{L}}{\partial\varphi^\alpha} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\right)\right]\delta\varphi^\alpha. \tag{8.120}$$

[17] This is true if the boundary ∂D_4 does not extend to spatial infinity; when the integration domain D_4 fills the whole space, we must require that the fields and their derivatives fall off sufficiently fast at infinity, or we may also use periodic boundary conditions. In any case the integration on an infinite domain can always be taken initially on a finite domain, and, after removing the boundary term, the integration domain can be extended to infinity.

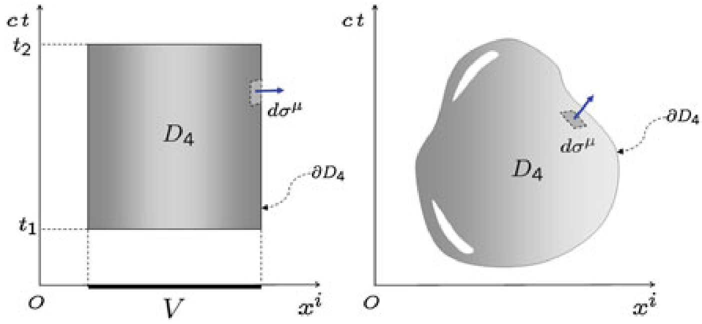

Fig. 8.1 Space–time domain D_4: of the form $[t_1, t_2] \times V$ (*left*), of generic form (*right*)

From the arbitrariness of $\delta\varphi^\alpha$ it follows:

$$\frac{\partial \mathcal{L}}{\partial \varphi^\alpha} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\alpha)} \right) = 0 \tag{8.121}$$

which are the Euler–Lagrange equations for the field φ^α, coinciding with (8.116).

As we have previously seen in the case of discrete dynamic systems, Lagrangians differing by a total time derivative lead to the same equations of motion. Similarly for field theories we can show that Lagrangian densities differing by a four-divergence $\partial_\mu f^\mu$ yield the same field equations. Indeed, let the Lagrangian densities $\mathcal{L}$ and $\mathcal{L}'$ be related by

$$\mathcal{L}'(\varphi^\alpha(x), \partial_\mu \varphi^\alpha(x), x) = \mathcal{L}(\varphi^\alpha(x), \partial_\mu \varphi^\alpha(x), x) + \partial_\mu f^\mu,$$

where $f^\mu = f^\mu(\varphi^\alpha(x), x)$, then the two actions differ by a boundary integral:

$$\begin{aligned} \mathcal{S}' &= \frac{1}{c} \int\limits_{D_4} d^4x \mathcal{L}'(\varphi^\alpha, \partial_\mu \varphi^\alpha) = \frac{1}{c} \int\limits_{D_4} d^4x \mathcal{L}(\varphi^\alpha, \partial_\mu \varphi^\alpha) \\ &+ \frac{1}{c} \int\limits_{D_4} d^4x \partial_\mu f^\mu = \mathcal{S} + \frac{1}{c} \int\limits_{\partial D_4} d^3\sigma_\mu f^\mu. \end{aligned} \tag{8.122}$$

We therefore have:

$$\delta\mathcal{S}' = \delta\mathcal{S} + \frac{1}{c} \int\limits_{\partial D_4} d^3\sigma_\mu \delta f^\mu = \delta\mathcal{S},$$

since $\delta f^\mu = \frac{\partial f^\mu}{\partial \varphi^\alpha(x)} \delta\varphi^\alpha(x) = 0$ on the boundary ∂D_4.

8.6 The Action of the Electromagnetic Field

As an application of our general discussion, we construct the action of the electromagnetic field in interaction with charges and currents and show that the stationary action principle gives the covariant form of the Maxwell equations discussed in Chap. 5. To this end we shall be guided by the symmetry principle. As it will be shown in detail in Sect. 8.7, the invariance of the equations of motion under space-time (i.e. Poincaré) transformations or under general field transformations is guaranteed if the Lagrangian density, as a function of the fields, their derivatives and the space-time coordinates, is invariant in form, up to a total divergence, see Eq. (8.150). As far as space-time translations are concerned, this is the case if $\mathcal{L}$ does not explicitly depend on x^μ. Covariance with respect to Lorentz transformations further requires $\mathcal{L}$ to be invariant as a function of the fields and their derivatives, namely to be a *Lorentz scalar* as a function of space-time.

The construction of the action for the electromagnetic field is relatively simple once we observe that:

- For the field $A_\mu(x)$ describing the electromagnetic field the generic index α coincides with the covariant index $\mu = 0, 1, 2, 3$ of the fundamental representation of the Lorentz group;
- The equations of motion (the Maxwell equations) are invariant under the *gauge transformations*:

$$A_\mu \to A_\mu + \partial_\mu \varphi.$$

  This is guaranteed if the Lagrangian density is invariant under the same transformations, since the action would then be invariant. In the absence of charges and currents, the action should be constructed out of the gauge invariant quantity $F_{\mu\nu}$;
- The Lagrangian density must be a *scalar* under Lorentz transformations;
- In order for the equations of motion to be second-order differential equations $\mathcal{L}$ must at most be quadratic in the derivatives of $A_\mu(x)$, that is quadratic in $F_{\mu\nu}$.

To construct Lorentz scalars which are quadratic in $F_{\mu\nu}$ we may use the invariant tensors $\eta_{\mu\nu}$, $\epsilon_{\mu\nu\rho\sigma}$ of the Lorentz group $\mathrm{SO}(1, 3)$.[18] It can be easily seen that the most general Lagrangian density satisfying the previous requirements has the following form:

$$\mathcal{L}(A_\mu, \partial_\mu A_\nu) = a F_{\mu\nu} F^{\mu\nu} + b \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}, \tag{8.123}$$

where $F^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma} F_{\rho\sigma}$ and a and b are numerical constants. On the other hand, the second term of (8.123) is the four-dimensional divergence of a four-vector so that it does not contribute to the equations of motion. Indeed:

$$\begin{aligned}
\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} &= 2\epsilon_{\mu\nu\rho\sigma} \partial^\mu A^\nu F^{\rho\sigma} \\
&= \partial^\mu \left(2\epsilon_{\mu\nu\rho\sigma} A^\nu F^{\rho\sigma}\right) - 2\epsilon_{\mu\nu\rho\sigma} A^\nu \partial^\mu F^{\rho\sigma} \\
&= \partial^\mu \left(2\epsilon_{\mu\nu\rho\sigma} A^\nu F^{\rho\sigma}\right) - 2\epsilon_{\mu\nu\rho\sigma} A^\nu \partial^{[\mu} F^{\rho\sigma]} \\
&= \partial_\mu f^\mu,
\end{aligned}$$

[18] Recall that the latter tensor $\epsilon_{\mu\nu\rho\sigma}$ is not invariant under Lorentz transformations which are in $\mathrm{O}(1, 3)$ but not in $\mathrm{SO}(1, 3)$, namely which have determinant -1. Examples of these are the parity transformation $\mathbf{\Lambda}_P$, or time reversal $\mathbf{\Lambda}_T$.

where we have set: $f^{\mu} = 2\epsilon^{\mu\nu\rho\sigma} A_{\nu} F_{\rho\sigma}$ and use has been made of the identity: $\partial^{[\mu} F^{\rho\sigma]} = 0$.

Therefore the Lagrangian density reduces, up to a four-dimensional divergence to the single term:

$$\mathcal{L}_{em} = a F_{\mu\nu} F^{\mu\nu}.$$

The value of the constant a is fixed in such a way that the Lagrangian contains the positive definite (density of) "kinetic term" $1/(2c^2)\ \partial_t A_i \partial_t A_i$ with a conventional factor $1/2$ which is remnant of the one appearing in the definition (8.25) of the kinetic energy.[19] Expanding $F_{\mu\nu} F^{\mu\nu} = (\partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu})(\partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu})$ one easily finds $a = -\frac{1}{4}$.

In the presence of charges and currents, the interaction with the source $J^{\mu}(x)$ requires adding an interaction term $\mathcal{L}_{int}$ to the pure electromagnetic Lagrangian. The simplest interaction is described by the Lorentz scalar term:

$$\mathcal{L}_{int} = b A_{\mu} J^{\mu}. \tag{8.124}$$

This term seems, however, to violate the gauge invariance of the total Lagrangian, since a gauge transformation on A_{μ} implies a correspondent change on the Lagrangian density:

$$\delta A_{\mu} = \partial_{\mu} \varphi \Rightarrow \delta_{(gauge)} \mathcal{L}_{int} = (\partial_{\mu} \varphi) J^{\mu}.$$

On the other hand by a partial integration $\delta \mathcal{L}_{int}$ can be transformed as follows:

$$\delta \mathcal{L}_{int} = \partial_{\mu}(\varphi J^{\mu}) - \varphi \partial_{\mu} J^{\mu}.$$

The first term is a total four-divergence, not contributing to the equations of motion and thus can be neglected; the second term is zero *if and only if* $\partial_{\mu} J^{\mu} = 0$, that is if the continuity equation expressing the conservation of the electric charge holds. We have thus found the following important result: *Requiring gauge invariance of the action of the electromagnetic field interacting with a current, implies the conservation of the electric charge.*

In conclusion, the action describing the electromagnetic field coupled to charges and currents is given by

$$S = \frac{1}{c} \int_{M_4} d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + b A_{\mu} J^{\mu} \right), \tag{8.125}$$

where the (four)-current $J^{\mu}(x)$ has the following general form (see Chap. 5)[20]:

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[19] Note that the kinetic term for A_0 is absent because of the antisymmetry of $F_{\mu\nu}$.

[20] Note that we are describing the interaction of the electromagnetic field, possessing infinite degrees of freedom, with a system of n charged particles, having $3n$ degrees of freedom represented by the n coordinate vectors $\mathbf{x}_{(k)}(t)$, $(k = i, \ldots, n)$. The Dirac delta function *formally* converts the $3n$ degrees of freedom of $\mathbf{x}_{(k)}(t)$ into the infinite degrees of freedom associated to $\mathbf{x}$.

$$J^\mu(x) = \frac{1}{c}\sum_k e_k \frac{dx_k^\mu}{dt}\delta^3(\mathbf{x} - \mathbf{x}_k(t)). \tag{8.126}$$

We may now apply the principle of stationary action to compute the equations of motion. Recalling the form (8.121) of the Euler–Lagrange equations for fields, we have:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\rho\left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\mu)}\right) = 0. \tag{8.127}$$

The first term of (8.127) is easily computed and gives:

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = bJ^\mu(x).$$

As far as the second term is concerned, only the pure electromagnetic part $-1/4F_{\mu\nu}F^{\mu\nu}$ contributes to the variation, yielding:

$$\frac{\partial(F_{\rho\sigma}F^{\rho\sigma})}{\partial(\partial_\mu A_\nu)} = 2\left[\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)}\right]F^{\rho\sigma} = 4\frac{\partial(\partial_\rho A_\sigma)}{\partial(\partial_\mu A_\nu)}F^{\rho\sigma} = \delta^\mu_\rho\delta^\nu_\sigma F^{\rho\sigma} = 4F^{\mu\nu}(x).$$

Putting together these results, (8.127) becomes:

$$\partial_\mu F^{\mu\nu}(x) + bJ^\nu(x) = 0. \tag{8.128}$$

Finally he constant b is fixed by requiring (8.128) to be identical to the Maxwell equation[21]:

$$\partial_\mu F^{\mu\nu} = -J^\nu,$$

and this fixes b to be 1. The final expression of the Lagrangian density therefore is:

$$\mathcal{L} = \mathcal{L}_{em} + \mathcal{L}_{int} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J^\mu. \tag{8.129}$$

In order to give a complete description of the charged particles in interaction with the electromagnetic field, we must add to $\mathcal{L}$ (8.129) the Lagrangian density $\mathcal{L}_{part}$ associated with system of particles.

Let us consider for the sake of simplicity the case of a single particle of charge e and mass m. The total action will have the following form[22]:

$$\mathcal{S}_{tot} = \mathcal{S}_{em} + \mathcal{S}_{int} + \mathcal{S}_{part}, \tag{8.130}$$

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[21] Note that this condition just fixes the charge normalization.

[22] The index k given to $\mathbf{x}_{(k)}$ in the following formulae has a double function: On the one hand it indicates that the coordinate vector $\mathbf{x}_{(k)}(t)$ is a *dynamic variable*, and not the labeling of the space points, as is the case for $\mathbf{x}$; On the other hand if we have more than one particle the following formulae can be generalized by just summing over k.

where

$$\mathcal{S}_{em}[\partial_\mu A_\nu] = \frac{1}{c}\int d^4x\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right),$$

$$\mathcal{S}_{part}[\dot{\mathbf{x}}_{(k)}] = -mc^2\int dt\left(1-\frac{v_{(k)}^2}{c^2}\right)^{\frac{1}{2}},$$

$$\begin{aligned}\mathcal{S}_{int}[A_\mu(x),\mathbf{x}_{(k)},\dot{\mathbf{x}}_{(k)}] &= \frac{1}{c}\int d^4x A_\mu(\mathbf{x},t)J^\mu(\mathbf{x},t)\\ &= \frac{1}{c}\int\left[\frac{e}{c}A_\mu(\mathbf{x}_{(k)},t)\frac{dx^\mu_{(k)}}{dt}\right]dt,\end{aligned} \tag{8.131}$$

where in deriving the expression of $\mathcal{S}_{int}$[23] we have used the explicit form of the four-current given in (8.126).

$$\begin{aligned}L_{int} &= \int d^3\mathbf{x}A_\mu(\mathbf{x},t)J^\mu(\mathbf{x},t) = \frac{e}{c}A_\mu(\mathbf{x}_{(k)},t)\frac{dx^\mu_{(k)}}{dt}\\ &= eA_0(\mathbf{x}_{(k)},t) + \frac{e}{c}A_i(\mathbf{x}_{(k)},t)v^i.\end{aligned} \tag{8.132}$$

We recall that $\mathbf{x}$ are labels of the points in space, while $\mathbf{x}_{(k)}(t)$ are the particle coordinates, that is dynamic variables, as stressed in the footnote.

We now observe that since $\mathcal{S}_{em}$ does not contain the variables $x^i_{(k)}$, we may compute the equation of motion of the charged particle by varying only $\hat{L} = L_{part} + L_{int}$:

$$\hat{L} = L_{part} + L_{int} = -mc^2\sqrt{1-\frac{v_{(k)}^2}{c^2}} + eA_0(\mathbf{x}_{(k)},t) + \frac{e}{c}A_i(\mathbf{x}_{(k)},t)v^i.$$

For the sake of simplicity in the following we neglect the index (k) of the particle. The first term of the Euler–Lagrange equations:

$$\frac{\partial\hat{L}}{\partial x^i} - \frac{d}{dt}\frac{\partial\hat{L}}{\partial v^i} = 0, \tag{8.133}$$

reads:

$$\frac{\partial\hat{L}}{\partial x^i} = \frac{\partial L_{int}}{\partial x^i} = e\frac{\partial A_0}{\partial x^i} + \frac{e}{c}\left(\frac{\partial A_j}{\partial x^i}\right)v^j. \tag{8.134}$$

The second term contains the time derivative of the *canonical momentum* p^i *conjugate to* x^i, namely:

[23] Note that also $\mathcal{S}_{part}$ can be written as a four-dimensional integral:
$$\mathcal{S}_{int} = -mc\int d^4x\delta^{(3)}(\mathbf{x}-\mathbf{x}_k)\left(1-\frac{1}{c^2}\left(\frac{d\mathbf{x}}{dt}\right)^2\right).$$

$$p^i = \frac{\partial \hat{L}}{\partial v^i} = \frac{\partial (L_{par} + L_{int})}{\partial v^i} = m(v)v^i + \frac{e}{c} A_i. \tag{8.135}$$

We see that in the presence of the electromagnetic field *the canonical conjugate momentum is different from the momentum* $p^i_{(0)} = m(v)v^i$ *of a free particle.*[24] In fact we have the following relation:

$$p^i = p^i_{(0)} + \frac{e}{c} A_i. \tag{8.136}$$

Taking into account (8.133), (8.135) and (8.136), the equation of motion of the charged particle becomes:

$$\frac{d}{dt}\left(p^i_{(0)} + \frac{e}{c} A_i\right) - e\partial_i A_0 - \frac{e}{c}\partial_i A_j v^j = 0. \tag{8.137}$$

We now recall that $A_0 = -V$, where V is the electrostatic potential. Moreover, since

$$\frac{dA_i}{dt} = \frac{\partial A_i}{\partial x^j}\frac{dx^j}{dt} + c\frac{\partial A_i}{\partial x^0},$$

and $E^i = F_{i0} = \partial_i A_0 - \partial_0 A_i$, (8.137) becomes:

$$\begin{aligned} \frac{dp^i_{(0)}}{dt} &= eE^i - \frac{e}{c}\left(\partial_j A_i - \partial_i A_j\right) v^j \\ &= eE^i - \frac{e}{c} F_{ji} v^j = eE^i + \frac{e}{c}\epsilon_{ijk} v^j B_k \\ &= e\left(E^i + \frac{1}{c}(\mathbf{v} \times \mathbf{B})^i\right). \end{aligned}$$

Thus we have retrieved from the variational principle the well known equation of motion of a charged particle subject to electric and magnetic fields since the right hand side is by definition the Lorentz force.

8.6.1 The Hamiltonian for an Interacting Charge

As we have computed the Lagrangian $L_{int} + L_{par}$ for a charged particle, we pause for a moment with our treatment of the Lagrangian formalism in field theories and compute the Hamiltonian of a charge interacting with the electromagnetic field. From the definition (8.64) we find[25]:

[24] Here and in the following we use the subscript 0 to denote the usual free-particle momentum $p^i_{(0)} = m(v)v^i$ and the symbol p^i for the momentum canonically conjugated to x^i.

[25] Recall that the vector $\mathbf{A} \equiv (A_i)$ is the spatial part of the four-vector $A_\mu \equiv (A_0, \mathbf{A})$, so that $A^\mu \equiv (A_0, -\mathbf{A})$. On the other hand $\mathbf{p}$ is the spatial component of $p^\mu \equiv (p^0, \mathbf{p})$, so that $p_\mu \equiv (p^0, -\mathbf{p})$.

$$H(\mathbf{p}, \mathbf{x}) = \mathbf{p} \cdot \mathbf{v} - L_{int} - L_{par} = \mathbf{p} \cdot \mathbf{v} - eA_0 - \frac{e}{c}\mathbf{A} \cdot \mathbf{v} + \frac{m^2c^2}{m(v)},$$

where we have used the relation

$$-L_{part} = mc^2\sqrt{1 - \frac{v^2}{c^2}} = \frac{m^2c^2}{m(v)}.$$

It follows:

$$H(\mathbf{p}, \mathbf{x}) = \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) \cdot \mathbf{v} + \frac{m^2c^2}{m(v)} + eV(\mathbf{x}). \tag{8.138}$$

We now use (8.135) to express v^i in terms of p^i:

$$\mathbf{v} = \frac{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)}{m(v)} = \frac{\mathbf{p}_{(0)}}{m(v)}.$$

Taking into account the relativistic relations:

$$E^2 = |\mathbf{p}_{(0)}|^2c^2 + m^2c^4; \quad m(v) = E/c^2, \tag{8.139}$$

where E is the energy of the free particle, we can write:

$$H(\mathbf{p}, \mathbf{x}) = \frac{E^2}{m(v)c^2} + eV = c\sqrt{m^2c^2 + \left|\mathbf{p} - \frac{e}{c}\mathbf{A}\right|^2} + eV(\mathbf{x}). \tag{8.140}$$

From the above equation we find:

$$(H + eA_0)^2 - c^2\sum_{i=1}^{3}\left(p^i - \frac{e}{c}A_i\right)^2 = m^2c^4. \tag{8.141}$$

Next we use the property $A^0 = A_0, \ A^i = -A_i$ to put (8.141) in relativistic invariant form:

$$\left(p^\mu + \frac{e}{c}A^\mu\right)\left(p_\mu + \frac{e}{c}A_\mu\right) = m^2c^2. \tag{8.142}$$

where we have set $\frac{H}{c} = p^0$.

Note that (8.142) can be obtained from the relativistic relation $p^\mu_{(0)}p_{(0)\mu} = m^2c^2$ of a free particle through the substitution:

$$p^\mu_{(0)} \to p^\mu + \frac{e}{c}A^\mu. \tag{8.143}$$

in agreement with (8.136). This substitution gives the correct coupling between the electromagnetic field and the charged particle and is usually referred to as *minimal coupling*.

8.7 Symmetry and the Noether Theorem

In this section we explore the connection between symmetry transformation and conservation laws in field theory.

We consider a relativistic theory described by an action of the following form:

$$S\left[\varphi^{\alpha}, D_4\right] = \frac{1}{c}\int_{D_4} d^4x\mathcal{L}(\varphi^{\alpha}, \partial_{\mu}\varphi^{\alpha}, x^{\mu}). \tag{8.144}$$

where $\mathcal{L}(\varphi^{\alpha}, \partial_{\mu}\varphi^{\alpha}, x)$ is the Lagrangian density.

We consider a generic transformation of the coordinates x^{μ} and of the fields φ^{α}:

$$\begin{aligned} x^{\mu} \in D_4 \to x'^{\mu} &= x'^{\mu}(x) \in D'_4, \\ \varphi^{\alpha} \to \varphi'^{\alpha} &= \varphi'^{\alpha}(\varphi^{\alpha}, x), \\ \partial_{\mu}\varphi^{\alpha} \to \partial_{\mu'}\varphi'^{\alpha} &= \partial'_{\mu}\varphi'^{\alpha}(\varphi^{\alpha}, \partial_{\mu}\varphi^{\alpha}, x). \end{aligned} \tag{8.145}$$

where $\partial'_{\mu} = \frac{\partial}{\partial x'^{\mu}}$. A transformation on space–time coordinates will in general deform a domain D_4, which we had originally taken to be a direct product of a time interval and a space volume V, into a region D'_4 with a different shape.

As already discussed in the case of a discrete set of degrees of freedom the actual value of the action computed on a generic four-dimensional domain D_4 does not depend on the set of fields and coordinates we use, since it is a *scalar*; In other words:

$$\mathcal{S}'\left[\varphi'^{\alpha}; D'_4\right] = \mathcal{S}\left[\varphi^{\alpha}; D_4\right], \tag{8.146}$$

or, more explicitly

$$\frac{1}{c}\int_{D'_4} d^4x'\mathcal{L}(\varphi'^{\alpha}(x'), \partial'_{\mu}\varphi'^{\alpha}(x'), x') = \frac{1}{c}\int_{D_4} d^4x\mathcal{L}(\varphi^{\alpha}(x), \partial_{\mu}\varphi^{\alpha}(x), x), \tag{8.147}$$

where the transformed Lagrangian density $\mathcal{L}'$ in $\mathcal{S}'$ is given by

$$\mathcal{L}'(\varphi'^{\alpha}, \partial'_{\mu}\varphi'^{\alpha}, x') = \mathcal{L}(\varphi^{\alpha}, \partial_{\mu}\varphi^{\alpha}, x), \tag{8.148}$$

the transformed fields and coordinates being related to the old ones by (8.145). However, as we have already emphasized in the case of a discrete system, the fact the action is a scalar, does not imply that the Euler–Lagrange equations derived from $\mathcal{S}$ ed $\mathcal{S}'$ have the same *form*. The latter property holds only when the transformations (8.145) correspond to an *invariance (or symmetry)* of the system. This is the case when the action is invariant, namely when:

$$S\left[\varphi'^{\alpha}; D'_4\right] = S\left[\varphi^{\alpha}; D_4\right]. \tag{8.149}$$

Note that (8.149) implies that the Lagrangian $\mathcal{L}$ is *invariant* under the transformations (8.145) only up to the four-divergence of an arbitrary four-vector f^μ, which, as we know, does not change the equations of motion:

$$\mathcal{L}(\varphi'^\alpha(x'), \partial'_\mu\varphi'^\alpha(x'), x') = \mathcal{L}(\varphi^\alpha(x), \partial_\mu\varphi^\alpha(x), x) + \partial_\mu f^\mu, \tag{8.150}$$

where $f^\mu = f^\mu(\varphi^\alpha(x), x)$.[26]

In the sequel we shall consider transformations differing by an infinitesimal amount from the identity, to which they are connected with continuity. We write these transformations in the following form:

$$\begin{aligned} x'^\mu &= x^\mu + \delta x^\mu, \\ \varphi'^\alpha(x) &= \varphi^\alpha(x) + \delta\varphi^\alpha(x), \end{aligned} \tag{8.151}$$

where δx^μ and $\delta\varphi^\alpha(x)$ are infinitesimals and, just as we did in Chap. 7, we define the *local* variation of the field as the difference $\delta\varphi^\alpha(x) \equiv \varphi'^\alpha(x) - \varphi^\alpha(x)$ between the transformed and the original fields evaluated in the same values of the coordinates $x = (x^\mu)$, see for instance (8.72). The invariance of the action under infinitesimal transformations is expressed by the equation:

$$c\delta\mathcal{S} = \int\limits_{D'_4} d^4x'\mathcal{L}(\varphi'^\alpha(x'), \partial'_\mu\varphi'^\alpha(x'), x') - \int\limits_{D_4} d^4x\mathcal{L}(\varphi^\alpha(x), \partial_\mu\varphi^\alpha(x), x) = 0, \tag{8.152}$$

where, for the time being, we do not consider the contribution a four-divergence $\partial_\mu f^\mu$ since it leads to equivalent actions.[27] The *Noether theorem* states that:

If the action of a physical system described by fields is invariant under a group of continuous global transformations, it is possible to associate with each parameter θ_r of the transformation group a four-current J_r^μ obeying the continuity equation $\partial_\mu J_r^\mu = 0$, and, correspondingly, a conserved charge Q_r, where

$$Q_r = \int d^3\mathbf{x} J_r^0. \tag{8.153}$$

Here by *global transformations* we mean transformations whose parameters do not depend on the space–time coordinates x^μ.

The proof of the theorem requires working out the consequences of (8.152) along the same lines as for the proof of the analogous theorem for systems with a finite number of degrees of freedom. For the sake of clarity we shall give, at each step of the proof, the reference to the corresponding formulae of Sect. 8.2.1.

[26] We note that the invariance of the action means that two configurations $[\varphi^\alpha(x), x^\mu \in D_4]$ and $\left[\varphi'^\alpha(x'), x'^\mu \in D'_4\right]$ related by the transformation (8.145) are solutions to the same partial differential equations.

[27] This freedom will be taken into account when discussing the energy momentum tensor in the next section.

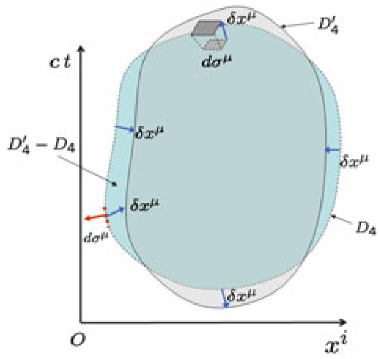

Fig. 8.2 Space–time domains D_4 and D'_4

We begin by observing that since x' is an integration variable, we may rewrite $\delta\mathcal{S}$ as follows (cfr. (8.47)):

$$c\delta\mathcal{S} = \int\limits_{D'_4} d^4x\mathcal{L}(\varphi'^{\alpha}(x), \partial_\mu\varphi'^{\alpha}(x), x) - \int\limits_{D_4} d^4x\mathcal{L}(\varphi^{\alpha}(x), \partial_\mu\varphi^{\alpha}(x), x). \quad (8.154)$$

The integration domains of the two integrals of (8.154) are D'_4 and D_4, respectively. In the discrete case we had $[t'_1, t'_2]$ and $t_1, t_2]$ instead of D'_4 and D_4. It is then convenient to write the first integral over D'_4 as the sum of an integral over D_4 and an integral over the "difference" $D'_4 - D_4$ between the two domains:

$$\int\limits_{D'_4} = \int\limits_{D_4} + \int\limits_{D'_4 - D_4} . \quad (8.155)$$

The domain $D'_4 - D_4$, see Fig. 8.2, can be decomposed in infinitesimal four-dimensional *hypercubes* having as basis the three-dimensional elementary volume $d\sigma$ on the boundary hypersurface ∂D_4 and height given by the elementary shift δx^μ of a point on $d\sigma$ due to the transformation (8.145). We have moreover defined $d\sigma^\mu \equiv n^\mu d\sigma$ as explained after (8.122).

Thus we may write an elementary volume in $D'_4 - D_4$ as follows:

$$d^4x = d\sigma_\mu \delta x^\mu,$$

so that the first integral on the right hand side of (8.154) reads

$$\int\limits_{D'_4} d^4x(\cdots) = \int\limits_{D_4} d^4x(\cdots) + \int\limits_{D'_4 - D_4} (\cdots)d^4x = \int\limits_{D_4} d^4x(\cdots) + \int\limits_{\partial D_4} d\sigma_\mu \delta x^\mu(\cdots). \quad (8.156)$$

A comparison with the analogous decomposition made in the discrete case, (8.48), reveals that ∂D_4 plays the role of the boundary of the interval $t_1 - t_2$ (consisting of the two end-points) and δx^μ of δt.

We may now insert this decomposition in (8.154) obtaining (see (8.51)):

$$c\delta\mathcal{S} = \int_{D_4} d^4x\mathcal{L}(\varphi'^\alpha(x), \partial_\mu\varphi'^\alpha(x), x) - \int_{D_4} d^4x\mathcal{L}(\varphi^\alpha(x), \partial_\mu\varphi^\alpha(x), x)$$
$$+ \int_{\partial D_4} d\sigma_\mu\delta x^\mu\mathcal{L}(\varphi^\alpha(x), \partial_\mu\varphi^\alpha(x), x), \tag{8.157}$$

where, in the last integral, we have replaced $\mathcal{L}(\varphi'^\alpha(x), \partial_\mu\varphi'^\alpha(x), x)$ with $\mathcal{L}(\varphi^\alpha(x), \partial_\mu\varphi^\alpha(x), x)$, since their difference, being multiplied by δx^μ would have been an infinitesimal of higher order (see the analogous equation (8.54)). On the other hand the difference between the first two integrals can be written as follows: (8.52):

$$\int_{D_4} d^4x\left[\mathcal{L}(\varphi'^\alpha(x), \partial_\mu\varphi'^\alpha(x), x) - \mathcal{L}(\varphi^\alpha(x), \partial_\mu\varphi^\alpha(x), x)\right]$$
$$= \int_{D_4} d^4x\left[\frac{\partial\mathcal{L}}{\partial\varphi^\alpha}\delta\varphi^\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\delta\partial_\mu\varphi^\alpha\right]$$
$$= \int_{D_4} d^4x\left[\frac{\partial\mathcal{L}}{\partial\varphi^\alpha} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\right]\delta\varphi^\alpha(x) + \int_{D_4} d^4x\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\delta\varphi^\alpha\right). \tag{8.158}$$

where, as usual, we have applied the property

$$\delta(\partial_\mu\varphi^\alpha) = \partial_\mu\delta\varphi^\alpha.$$

Finally we substitute (8.157) and (8.158) into (8.152) obtaining, for the variation of the action (see (8.53)):

$$c\delta\mathcal{S} = \int_{D_4} d^4x\left[\frac{\partial\mathcal{L}}{\partial\varphi^\alpha} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\right]\delta\varphi^\alpha(x) + \int_{D_4} d^4x\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\delta\varphi^\alpha\right)$$
$$+ \int_{\partial D_4} d\sigma_\mu\delta x^\mu\mathcal{L}(\varphi^\alpha(x), \partial_\mu\varphi^\alpha(x), x). \tag{8.159}$$

If the Euler–Lagrange equations (8.121) are satisfied, the first integral in (8.159) vanishes; moreover the last integral can be written as an integral on ∂D_4 by use of the four-dimensional Gauss (or divergence theorem) theorem in reverse:

$$\int_{\partial D_4} d\sigma_\mu\delta x^\mu\mathcal{L} = \int_{D_4} d^4x\partial_\mu(\delta x^\mu\mathcal{L}). \tag{8.160}$$

We have thus obtained:

$$\delta\mathcal{S} = \frac{1}{c}\int_{D_4} d^4x\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\delta\varphi^\alpha + \delta x^\mu\mathcal{L}\right]. \tag{8.161}$$

The above equation gives the desired result: it states that when $\delta\mathcal{S} = 0$, the integral in (8.161) is zero. Taking into account that the integration domain is arbitrary, we must have:

$$\partial_\mu J^\mu = 0, \tag{8.162}$$

where

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\delta\varphi^\alpha + \delta x^\mu\mathcal{L}. \tag{8.163}$$

In terms of the infinitesimal, global parameters $\delta\theta^r, r = 1, \ldots, g$ of the continuous transformation group G, the infinitesimal variations $\delta\varphi^\alpha$ and δx^μ can be written as

$$\delta\varphi^\alpha = \delta\theta^r\Phi^\alpha_r; \quad \delta x^\mu = \delta\theta^r X^\mu_r. \tag{8.164}$$

where Φ^α_r andX^μ_r are, in general, functions of the fields φ^α and coordinates x^μ. Thus we may write:

$$J^\mu = \delta\theta^r J^\mu_r,$$

where

$$J^\mu_r = \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\Phi^\alpha_r + X^\mu_r\mathcal{L}\right). \tag{8.165}$$

Taking into account that the $\delta\theta^r$ are independent, constant parameters, we can state that we have a set of g conserved currents $\partial_\mu J^\mu_r = 0$. To each conserved current J^μ_r there corresponds a conserved charge Q_r:

$$Q_r = \int_{\mathbb{R}^3} d^3\mathbf{x} J^0_r, \tag{8.166}$$

where we take as V the entire three-dimensional space $\mathbb{R}^3$. Indeed:

$$\frac{dQ_r}{dt} = c\int_{\mathbb{R}^3} d^3\mathbf{x}\frac{\partial}{\partial x^0}J^0_r = -\int_{\mathbb{R}^3} d^3\mathbf{x}\frac{\partial}{\partial x^i}J^i_r = -\int_{S_\infty} d^2\sigma\sum_{i=1}^3 J^i_r n^i = 0.$$

where the last surface integral is zero being evaluated at infinity where the currents are supposed to vanish.

8.8 Space–Time Symmetries

As already stressed in the first Chapters of this book, in order to satisfy the principle of relativity a physical theory must fulfil the requirement of *invariance under the Poincaré group*. The latter was discussed in detail in Chap. 4 and contains, as subgroups, the Lorentz group and the four-dimensional translation group. Invariance of a theory, describing an isolated system of fields, under Poincaré transformations implies that its predictions cannot depend on a particular direction or on a specific space–time region in which we observe the system, consistently with our assumption of *homogeneity and isotropy* of Minkowski space.

The Noether theorem allows us to derive conservation laws as a consequence of this invariance. Let us first work out the conserved charges associated with the invariance of the theory under *space–time translations*:

$$\begin{aligned} &x^\mu \to x'^\mu = x^\mu - \epsilon^\mu \longrightarrow \delta x^\mu = -\epsilon^\mu, \\ &\varphi'^\alpha(x-\epsilon) = \varphi^\alpha(x) \Rightarrow \delta\varphi^\alpha(x) = \varphi'^\alpha(x) - \varphi^\alpha(x) = \frac{\partial \varphi^\alpha(x)}{\partial x^\mu}\epsilon^\mu. \end{aligned} \tag{8.167}$$

Comparing this with the general formula (8.164) we can identify the index r with the space–time one μ, the parameters $\delta\theta^r$ with ϵ^μ and

$$\Phi^\alpha_r = \Phi^\alpha_\nu = \partial_\mu \varphi^\alpha \delta^\mu_\nu; \quad X^\mu_r = X^\mu_\nu = -\delta^\mu_\nu.$$

Requiring invariance of the action under the transformations (8.167), and inserting the values of δx^μ and $\delta\varphi^\alpha$ in the general expression of the current (8.163) we obtain:

$$J^\mu = J^\mu_\rho \epsilon^\rho = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\alpha)}\partial_\rho \varphi^\alpha - \delta^\mu_\rho \mathcal{L}\right)\epsilon^\rho \equiv c\epsilon^\rho T^\mu_\rho \tag{8.168}$$

where we have introduced the *energy-momentum*[28] tensor $T_{\mu|\rho}$:

$$T_{\mu|\rho} \equiv \frac{1}{c}\left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi^\alpha)}\partial_\rho \varphi^\alpha - \eta_{\mu\rho}\mathcal{L}\right], \tag{8.169}$$

so that we have the general conservation law:

$$\partial_\mu T^\mu_\nu = 0. \tag{8.170}$$

We note that both the indices of $T_{\mu|\nu}$ are Lorentz indices, but we have separated them by a bar since the first index is the index of the four-current while the second index is the index r labeling the parameters. This being understood, in the following we suppress the bar between the two indices of $T_{\mu|\nu}$.

[28] The alternative name of *stress-energy tensor* is also used.

The four Noether charges associated with the space–time translations are obtained by integration of $J^0_\mu \equiv cT^0_\mu$ over the whole three-dimensional space.

$$Q_\mu = c\int d^3\mathbf{x}T_{0\mu} \doteq cP_\mu, \tag{8.171}$$

and, from the Noether-conservation law $\partial_\nu T^{\nu\mu}$, we obtain in the usual way that:

$$\frac{d}{dt}P^\mu = 0.$$

To understand the physical meaning of the energy-momentum tensor and of the conserved four-vector

$$P^\mu \equiv \int\limits_V d^3\mathbf{x}\mathscr{P}^\mu \equiv \int\limits_V d^3\mathbf{x}T^{0\mu}, \tag{8.172}$$

where we have defined $\mathscr{P}^\mu \equiv T^{0\mu}$, we recall that in the case of systems with a finite number of degrees of freedom, the conserved four charges associated with space–time translations are the components of the four-momentum. It is natural then to interpret P^μ as the *total conserved four-momentum* associated with the continuous system under consideration, described by the fields $\varphi^\alpha(x)$.

As a consequence of this the tensor $T_{\mu\nu}$ can be thought of as describing the *density of energy and momentum and their currents in space and time*. In particular, $\mathscr{P}^\mu \equiv T^{0\mu}$ represents the *spatial density of the four-momentum*. We conclude that the *four conserved charges Q_μ/c associated with the space–time translations, which are an invariance of an isolated system, are the components of the total four-momentum.*

Let us now consider the further six conserved charges associated with the invariance with respect to Lorentz transformations.

Under such a transformation, the fields φ^α will transform according to the $SO(1,3)$ representation, labeled by the index α, which they belong to; its infinitesimal form has being given in (7.83), namely:

$$\delta\varphi^\alpha = \frac{1}{2}\delta\theta^{\rho\sigma}\left[(L_{\rho\sigma})^\alpha_\beta\varphi^\beta + (x_\rho\partial_\sigma - x_\sigma\partial_\rho)\varphi^\alpha\right]. \tag{8.173}$$

If the action is invariant under the Lorentz group, substitution of the variations (8.173) and (7.82) into (8.163) gives the following conserved current:

$$J_\mu = -\frac{c}{2}\delta\theta^{\rho\sigma}\mathcal{M}_{\mu|\rho\sigma}, \tag{8.174}$$

where we have introduced the tensor:

$$\mathcal{M}_{\mu|\rho\sigma} = -\frac{1}{c}\left[\frac{\partial\mathcal{L}}{\partial(\partial^\mu\varphi^\alpha)}\left((L_{\rho\sigma})^\alpha_\beta\varphi^\beta + (x_\rho\partial_\sigma - x_\sigma\partial_\rho)\varphi^\alpha\right)\right.$$
$$\left. + \left(x_\sigma\eta_{\mu\rho} - x_\rho\eta_{\mu\sigma}\right)\mathcal{L}\right], \tag{8.175}$$

and used the identification of the index r with the antisymmetric couple of indices $(\mu\nu)$ labeling the Lorentz generators, so that

$$X_r^\mu \equiv X_{\rho\sigma}^\mu = \delta_\rho^\mu x_\sigma - \delta_\sigma^\mu x_\rho,$$
$$\Phi_r^\alpha \equiv \Phi_{\rho\sigma}^\alpha = (L_{\rho\sigma})^\alpha_\beta \varphi^\beta + (x_\rho \partial_\sigma - x_\sigma \partial_\rho)\varphi^\alpha. \tag{8.176}$$

Comparing (8.175) with the definition of the *energy-momentum tensor* $T_{\mu\rho}$, (8.169), little insight reveals that the two terms proportional to x_μ within square brackets in the former can be expressed in terms of $T_{\mu\rho}$ as $x_\sigma T_{\mu\rho} - x_\rho T_{\mu\sigma}$. Therefore the conserved current $\mathcal{M}_{\mu|\rho\sigma}$ takes the simpler form:

$$\mathcal{M}_{\mu|\rho\sigma} = -\left[\frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi^\alpha)}(L_{\rho\sigma})^\alpha_\beta \varphi^\beta + (x_\rho T_{\mu\sigma} - x_\sigma T_{\mu\rho})\right]. \tag{8.177}$$

Being this current associated with Lorentz transformations which are always a symmetry of a relativistic theory, the Noether theorem implies:

$$\partial_\mu \mathcal{M}^\mu{}_{\rho\sigma} = 0. \tag{8.178}$$

Using the explicit form (8.177) in the conservation law (8.178) together with (8.170), we derive the following two equations:

$$\partial^\mu H_{\mu\rho\sigma} + T_{\rho\sigma} - T_{\sigma\rho} = 0, \tag{8.179}$$

$$\partial^\mu T_{\mu\nu} = 0. \tag{8.180}$$

where we have set

$$H_{\mu\rho\sigma} \equiv \frac{1}{c}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^\alpha)}(L_{\rho\sigma})^\alpha_\beta \varphi^\beta.$$

While (8.180) yields again the conservation law associated with the energy-momentum tensor, (8.179) implies a condition that cannot be satisfied if $T_{\mu\nu}$ is not symmetric in its two lower indices. To see this, let us consider the case of a scalar field $\varphi(x)$ carrying no representation index α so that $H_{\mu\rho\sigma} = 0$. Then $T_{\rho\sigma} - T_{\sigma\rho} \neq 0$ would be inconsistent with (8.179) which is a consequence of the Noether theorem (8.178). Actually, as it is apparent from its definition, $T^{\rho\sigma}$ *is in general not a symmetric tensor*, thus ruining the conservation law (8.178), which, as we shall show shortly, in particular implies the *conservation of the total angular momentum*. To solve this seeming inconsistency we note that the definition (8.169) does not determine the energy-momentum tensor uniquely. If we indeed redefine $T^{\mu\nu}$ as:

$$T^{\mu\nu} \to T^{\mu\nu} + \partial_\rho U^{\nu\mu\rho}, \quad U^{\nu\mu\rho} = -U^{\nu\rho\mu} \tag{8.181}$$

it still satisfies the conservation law, since $\partial_\mu \partial_\rho U^{\nu\mu\rho} \equiv 0$, due to the antisymmetry of $U^{\nu\mu\rho}$ in its last two indices. This possibility is related to the freedom we have of adding to the Lagrangian a four-divergence $\partial_\mu f^\mu$. Although we had neglected such freedom when proving the Noether theorem, one can exploit it to obtain a *symmetric energy-momentum tensor*.[29] To show this, let us perform the following redefinitions:

$$\Theta_{\mu\nu} = T_{\mu\nu} + \partial^\lambda U_{\nu\mu\lambda}; \quad U_{\nu\mu\lambda} = -U_{\nu\lambda\mu}, \tag{8.182}$$

$$\hat{\mathcal{M}}_{\mu|\rho\sigma} = \mathcal{M}_{\mu\rho\sigma} - \partial^\lambda \left(x_\rho U_{\sigma\mu\lambda} - x_\sigma U_{\rho\mu\lambda}\right). \tag{8.183}$$

where $\Theta_{\mu\nu}$ is the new energy momentum tensor. As already remarked these redefinitions do not spoil the conservation law associated with the energy-momentum tensor, since, due to the antisymmetry of $U_{\mu\nu\lambda}$ in the last two indices, we still have $\partial^\mu \Theta_{\mu\nu} = 0$. Moreover, by the same token, it is easily shown that $\hat{\mathcal{M}}_{\mu|\rho\sigma}$ is still conserved, i.e. $\partial^\mu \hat{\mathcal{M}}_{\mu|\rho\sigma} = 0$, since $\mathcal{M}_{\mu|\rho\sigma}$ is and the additional term $\partial^\lambda \left(x_\rho U_{\sigma\mu\lambda} - x_\sigma U_{\rho\mu\lambda}\right)$ is divergenceless by virtue of the antisymmetry of $U_{\sigma\mu\lambda}$ in its last two indices:

$$\partial^\mu \partial^\lambda \left(x_\rho U_{\sigma\mu\lambda} - x_\sigma U_{\rho\mu\lambda}\right) = 0. \tag{8.184}$$

Let us now show that $\hat{\mathcal{M}}_{\mu\rho\sigma}$ can be written in the simpler form:

$$\hat{\mathcal{M}}_{\mu|\rho\sigma} = -x_\rho \Theta_{\mu\sigma} + x_\sigma \Theta_{\mu\rho}, \tag{8.185}$$

by a suitable choice of $U_{\sigma\mu\lambda}$. If we prove this, then, from the conservation of the current $\hat{\mathcal{M}}_{\mu|\rho\sigma}$, we have

$$0 = \partial^\mu \hat{\mathcal{M}}_{\mu\rho\sigma} = -\delta^\mu_\rho \Theta_{\mu\sigma} + \delta^\mu_\sigma \Theta_{\mu\rho} = -\Theta_{\rho\sigma} + \Theta_{\sigma\rho}, \tag{8.186}$$

which implies that $\Theta_{\mu\nu}$ is symmetric. To prove (8.185) we first write the explicit form of $\hat{\mathcal{M}}_{\mu\rho\sigma}$ by expressing $T_{\mu\nu}$ in (8.177) in terms of $\Theta_{\mu\nu}$ and use the following identity:

$$-x_\rho \partial^\lambda U_{\sigma\mu\lambda} + x_\sigma \partial^\lambda U_{\rho\mu\lambda} = -\partial^\lambda \left(x_\rho U_{\sigma\mu\lambda} - x_\sigma U_{\rho\mu\lambda}\right) - U_{\rho\mu\sigma} + U_{\sigma\mu\rho}.$$

The four-divergence on the right hand side cancels against the opposite term in (8.183) and we end up with:

$$\hat{\mathcal{M}}_{\mu|\rho\sigma} = -H_{\mu\rho\sigma} - x_\rho \Theta_{\mu\sigma} + x_\sigma \Theta_{\mu\rho} - U_{\sigma\mu\rho} + U_{\rho\mu\sigma}. \tag{8.187}$$

Thus in order for $\hat{\mathcal{M}}$ to have the form (185) we need to find a tensor $U_{\nu\mu\lambda}$ satisfying the following condition:

$$U_{\rho\mu\sigma} - U_{\sigma\mu\rho} = \frac{1}{c}\frac{\partial\mathcal{L}}{\partial\partial^\mu\varphi^\alpha}(L_{\rho\sigma})^\alpha_\beta \varphi^\beta \equiv H_{\mu\rho\sigma}. \tag{8.188}$$

---

[29] We shall illustrate an application of this mechanism in the case of the electromagnetic field.

The solution is[30]

$$U_{\mu\rho\sigma} = \frac{1}{2}\left[H_{\mu\rho\sigma} - H_{\sigma\mu\rho} - H_{\rho\sigma\mu}\right]. \tag{8.189}$$

Let us now discuss the physical meaning of the conservation law (8.178), by computing the conserved "charges" $Q_{\rho\sigma}$ associated with the 0-component of the current $\mathcal{M}_{\mu|\rho\sigma}$ (8.186). Let us rename $Q_{\rho\sigma} \to J_{\rho\sigma}$, since, as we shall presently see, they are related to the angular momentum. Then integrating over the whole space $V = \mathbb{R}^3$:

$$J_{\rho\sigma} = \int_V d^3\mathbf{x} M_{0\rho\sigma} = -\int_V d^3\mathbf{x}\left[\frac{\partial\mathcal{L}}{\partial\dot{\varphi}^\alpha}(L_{\rho\sigma})^\alpha_\beta\varphi^\beta + (x_\rho T_{0\sigma} - x_\sigma T_{0\rho})\right], \tag{8.190}$$

are the conserved charged associated with Lorentz invariance:

$$\frac{d}{dt}J^{\rho\sigma} = 0. \tag{8.191}$$

In particular for spatial indices $(\mu\nu) = (ij)$ we find:

$$J_{ij} = -\int_V d^3\mathbf{x}\left[\frac{\partial\mathcal{L}}{\partial\dot{\varphi}^\alpha}(L_{ij})^\alpha_\beta\varphi^\beta + (x_i\mathscr{P}_j - x_j\mathscr{P}_i)\right] = -\epsilon_{ijk}J^k, \tag{8.192}$$

where $\mathscr{P}^i$ is the momentum density.

Let us first consider the case of a *scalar field* φ which, by definition, does not have internal components transforming under Lorentz transformations, so that the first term of (8.192) is absent. The second term in the integrand of (8.192) is easily recognized as the *density of orbital angular momentum*. Therefore $J_{ij} \equiv -\epsilon_{ijk}M^k$ is the conserved orbital angular momentum,which, for a scalar field, coincides with the total angular momentum.

If, however, we have a field φ^α transforming, through the index α, in a non-trivial representation of the Lorentz group, the first term in (8.177) is not zero; it is clear that it should also describe an angular momentum which must then refer to the intrinsic degrees of freedom of the field.[31] In fact the first term describes the *intrinsic angular momentum or spin* of the field.

In general if the field is not spinless the conservation law implies that only the sum of the orbital angular momentum and of the spin, that is only the *total angular momentum* is conserved.

Note that so far we have been discussing the conservation of the three charges J_{ij} associated with the invariance under three dimensional rotations and corresponding

[30] The solution (8.189) can be obtained writing, besides (8.188), two analogous equations obtained by cyclic permutation of the indices $\rho\mu\sigma$. Subtracting the last two equations from the first and using the antisymmetry property (8.181) we find (8.189).

[31] Recall from Chap. 4 that, since $L_{\rho\sigma}$ are Lorentz generators, $L_i = -\epsilon_{ijk}L_{jk}/2$ are generators of the rotation group.

to the components of the total angular momentum. It is interesting to understand the meaning of the other three conservation laws (8.178) associated with the invariance under Lorentz boosts, that is to the components J^{0i} of the $J^{\mu\nu}$ charges. Restricting for simplicity to the case of a scalar field, we have from (8.190) and (8.191), setting $(L_{ij})^\alpha_\beta = 0$,

$$\frac{d}{dt}\left[x^0\int d^3\mathbf{x}T^{0i} - \int x^i T^{00}d^3\mathbf{x}\right] = 0. \tag{8.193}$$

Taking into account the conservation of P^i, defined by the first integral, we obtain:

$$cP^i = \frac{d}{dt}\int x^i T^{00}d^3\mathbf{x}. \tag{8.194}$$

On the other hand since cT^{00} represents the energy density, we have $T^{00}d^3\mathbf{x} = dE/c = cdm$, where E is the total energy related to the total mass by the familiar relation $E = mc^2$. It then follows:

$$\mathbf{P} = \frac{d}{dt}\int \mathbf{x}dm. \tag{8.195}$$

In words: The conservation law associated with the Lorentz boosts implies that the *relativistic center of mass* moves at constant velocity.

8.8.1 Internal Symmetries

The symmetries and the associated conserved charges discussed in the previous section are space–time symmetries, namely symmetries associated with translations and Lorentz transformations under which, in a relativistic theory, the action is invariant.

We now want to give an example of a symmetry which does not involve changes in the space–time coordinates x^μ, but that is rather implemented by transformations acting on the internal index α of a field $\varphi^\alpha(x)$. In this case the index α labels the basis of a representation of the corresponding symmetry group G. Such symmetries are called *internal symmetries* and G is the internal symmetry group:

$$\begin{aligned} x^\mu \to x'^\mu = x^\mu &\Rightarrow \delta x^\mu = 0,\\ \varphi^\alpha(x) \to \varphi'^\alpha(x) &= \varphi^\alpha(x) + \delta\varphi^\alpha(x), \end{aligned} \tag{8.196}$$

where

$$\delta\varphi^\alpha = \delta\theta^r(L_r)^\alpha_\beta\varphi^\beta, \quad \delta\theta^r \ll 1.$$

From (8.163) the conserved currents have the simpler form:

$$\delta\theta^r J_r^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^\alpha)}\delta\varphi^\alpha. \tag{8.197}$$

The simplest, albeit important, example is the case in which we have *two* real scalar fields φ_1, φ_2, or, equivalently, a complex scalar field φ, φ^*, the two descriptions being related by

$$\varphi_1 = \frac{1}{\sqrt{2}}(\varphi + \varphi^*); \quad \varphi_2 = -\frac{i}{\sqrt{2}}(\varphi - \varphi^*),$$

and a Lagrangian density of the following form:

$$\mathcal{L} = c^2\left(\partial_\mu\varphi^*\partial^\mu\varphi - \frac{m^2c^2}{\hbar^2}\varphi^*\varphi\right). \tag{8.198}$$

The Euler–Lagrangian equations are

$$\hbar^2\partial_\mu\partial^\mu\varphi + m^2c^2\varphi = 0. \tag{8.199}$$

As will be shown in the next Chapter this equation is the natural *relativistic* extension of the Schrödinger equation for a particle of mass m and wave function φ. It is referred to as the *Klein–Gordon equation*, and the Lagrangian (8.198) is the *Klein–Gordon Lagrangian density*.

We observe that the Lagrangian density $\mathcal{L}$ of (8.198) is invariant under the following transformation:

$$\varphi(x) \to \varphi'(x) = e^{-i\alpha}\varphi(x) \tag{8.200}$$

where α is a constant parameter.

In the real basis, the transformation belongs to the group SO(2):

$$\begin{pmatrix}\varphi_1' \\ \varphi_2'\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix}. \tag{8.201}$$

In the complex basis the transformation (8.200) defines a one-parameter Lie group of unitary transformations denoted by $\mathbf{U}(1)$, which is isomorphic to, i.e. has the same structure as, $\mathbf{SO}(2)$. The infinitesimal version of (8.200) is

$$\varphi(x) \to \varphi'(x) \simeq \varphi(x) - i\alpha\varphi(x) \Rightarrow \delta\varphi(x) = -i\alpha\varphi(x); \quad \delta\varphi^* = i\alpha\varphi^*.$$

Using a suitable multiplicative coefficient to normalize the conserved current J^μ to the dimension of the electric current, we obtain from (8.197):

$$\alpha J^\mu = \frac{e}{c\hbar}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}\delta\varphi^*\right] = -i\frac{ec}{\hbar}\left[\varphi\partial^\mu\varphi^* - \varphi^*\partial^\mu\varphi\right]\alpha. \tag{8.202}$$

Let us verify the conservation law $\partial_\mu J^\mu = 0$ explicitly:

$$
\begin{aligned}
i\frac{\hbar}{ec}\partial_\mu J^\mu &= \partial_\mu\varphi\partial^\mu\varphi^* + \varphi\partial_\mu\partial^\mu\varphi^* - \partial_\mu\varphi^*\partial^\mu\varphi - \varphi^*\partial_\mu\partial^\mu\varphi \\
&= \frac{m^2c^2}{\hbar^2}\varphi\varphi^* - \frac{m^2c^2}{\hbar^2}\varphi\varphi^* = 0
\end{aligned}
$$

where we have used the equation of motion (8.199). We shall see in Chap. 10 that the conserved charge:

$$
Q = \int d^3\mathbf{x} J^0 = i\frac{e}{\hbar}\int d^3\mathbf{x}(\varphi^*\partial_t\varphi - \varphi\partial_t\varphi^*) \tag{8.203}
$$

can be identified with *electric charge* of a scalar field φ interacting with the electromagnetic field.

Let us note that if the field were real, $\varphi(x) = \varphi^*(x)$, that is if we had just one field, there would be no invariance of the Lagrangian and the charge Q would be zero. As it will be shown in the sequel, this is a general feature: when a field is interpreted as the wave function of a particle, a *real* field describes a neutral particle, as it happens for the photon field $A_\mu(x) = A^*_\mu(x)$, while fields associated with charged particles are intrinsically *complex*.

8.9 Hamiltonian Formalism in Field Theory

In the previous section we have described systems with a continuum of degrees of freedom using the Lagrangian formalism. We want now discuss the dynamics of such systems using the Hamiltonian formalism. The most direct way to derive the Hamiltonian description of field dynamics is to use the limiting procedure discussed in Sect. 8.5.1 for the Lagrangian formalism

Consider a theory describing a field $\varphi(x)$ (let us suppress the internal index α for the time being). Just as we did in Sect. 8.5.1, we divide the three-dimensional domain V in which we study the system, into tiny cells of volume δV^i, defining the Lagrangian coordinates $\varphi_i(t)$ as the mean value of $\varphi(\mathbf{x}, t)$ within the ith cell. We thus have a discrete dynamic system and define the momenta p_i conjugate to φ_i as

$$
p_i = \frac{\partial L(t)}{\partial\dot{\varphi}_i(t)}. \tag{8.204}
$$

The Hamiltonian of the system is given by

$$
H = \sum_i p_i\dot{\varphi}_i - L, \tag{8.205}
$$

with equations of motion:

$$
\dot{\varphi}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial\varphi_i}. \tag{8.206}
$$

Recall now from the discussion in Sect. 8.5.1 that, in the continuum limit (δV_i infinitesimal)